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RAYS AND CAUSTICS IN VERTICALLY INHOMOGENEOUS ELASTIC MEDIA

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN GEOPHYSICS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ..RAYS AND CAUSTICS.... ...IN VERTICALLY INHOMOGENEOUS ELASTIC MEDIA............ .... submitted by ....A. P. CHOI..................................... in partial fulfilment of the requirements for the degree of Master of Science in Geophysics.

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ABSTRACT

Synthetic seismograms have become an essential tool in the interpretation of real seismic records. Due to the low cost of numerical computation and the possibility to identify and study individual events on seismograms, asymptotic ray theory (ART) is well suited for this purpose.

Asymptotic ray theory, however, has its limitations. It can neither be used for the investigation of diffracted waves in shadow zones, nor in the neighborhood of singular points such as caustics and critical points.

The aim of this thesis is to extend the applicability of ART to include the effects of caustics. First, the essential features of ART are presented. Then, using the wave method, a formal integral solution for an arbitrary ray in a layered, vertically inhomogeneous medium is derived. Evaluating this integral solution using a modified version of the third-order saddle point approximation, a new expression is derived for the amplitude in the vicinity of a caustic. This result is shown to be more accurate than the amplitude expression currently in common use. Pulse distortion due to caustics is also investigated and a new expression is derived allowing easy determination of the phase shift. Finally, numerical results (synthetic seismograms, ray diagrams, time-distance and amplitude-distance curves) based on the extended ART are presented.

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CHAPTER 1

ASYMPTOTIC RAY THEORY FOR VERTICALLY INHOMOGENEOUS MEDIA

This chapter presents the basic results of ART applied to the problem of elastic wave propagation. Details of the theory can be found in Karal and Keller (1958), Hron and Kanasewich (1971), and Cerveny and Ravindra (1971).

The linearized equations of motion governing small amplitude wave propagation in an inhomogeneous, perfectly elastic and isotropic medium are

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) + \mu\nabla^2\vec{u} + \nabla\lambda(\nabla\cdot\vec{u}) + \nabla\mu\times(\nabla\times\vec{u}) + 2(\nabla\mu\cdot\nabla)\vec{u} \qquad 1.1$$

where $\vec{u}$ is the particle displacement vector, t is the time, ρ is the density of the medium and λ and μ its Lamé parameters.

The ray method assumes that the complete solution to 1.1 and given boundary conditions can be decomposed into an infinite number of individual contributions each of which is independently a solution of the problem. It is further assumed that an individual contribution can be expanded in an asymptotic series,

called the ray series, in inverse powers of the frequency

$$\vec{u} = S(\omega)\ e^{-i\omega\tau} \sum_{k=0}^{\infty} (i\omega)^{-k}\vec{u}_k$$

Here $S(\omega)$ is the spectrum of the wave pulse, τ and $\vec{u}_k$, which are functions only of the spatial coordinates, are called the phase function and amplitude coefficients of the ray series.

The moving surfaces of constant phase, $t = \tau(x,y,z)$, are called wave fronts and the orthogonal trajectories of the surfaces rays. In general, any lit point in the medium is uniquely defined by the position vector $\vec{X} = \vec{X}(\tau, q_1, q_2)$ where the curvilinear coordinates q_1 and q_2, called the ray coordinates, characterize a ray while the phase τ defines the position of a point along the ray. By a ray tube we will mean the family of rays the parameters of which are within the limits (q_1, q_1+dq_1) and (q_2, q_2+dq_2). On any given wavefront the cross-sectional area $d\sigma$ of the ray tube is

$$d\sigma = \left| \frac{\partial\vec{X}}{\partial q_1} \times \frac{\partial\vec{X}}{\partial q_2} \right| dq_1\ dq_2 \qquad 1.2$$

Due to the inverse frequency dependence of the ray series, it is apparent that the first term is the most significant. In this thesis, only the first term of the

ray series will be considered. This leads to the zeroth-order solution or zeroth approximation of ART:

$$\vec{u} = S(\omega)\ e^{-i\omega\tau}\vec{U}(x,y,z)$$

Furthermore, it suffices to consider only an arbitrary ray as similar methods can be applied to compute all other rays composing the total solution.

In vertically inhomogeneous media all rays are plane curves. Thus a Cartesian coordinate can be adopted so that the entire ray is contained in the vertical xz-plane. Then three orthogonal unit vectors $\vec{n}_p$, $\vec{n}_{SV}$ and $\vec{n}_{SH}$ can be introduced at every point of the ray such that $\vec{n}_p$ is tangent to the ray pointing in the direction of propagation; $\vec{n}_{SV}$ is perpendicular to the ray and lies in the vertical plane such that its projection on the x-axis is positive; and $\vec{n}_{SH}$ is a unit vector in the direction of the y-axis.

Substituting the zeroth-order solution into the equations of motion one finds that

$$\vec{u} = S(\omega)\ e^{-i\omega\tau}\ U_i\vec{n}_i$$

where $\vec{n}_i$ may be $\vec{n}_p$, $\vec{n}_{SV}$ or $\vec{n}_{SH}$, and U_i are the corresponding amplitude coefficients. This means that

in the zeroth order approximation of ART three kinds of elastic waves, P waves, SV and SH waves, can propagate within vertically inhomogeneous media.

In all three cases, the phase function τ satisfies the well known eikonal equation (see e.g. Cerveny and Ravindra 1971, eq. 2.15 and 2.16) which can be solved to yield

$$\tau(M) = \tau(M_o) + \int_{M_o}^{M} \frac{ds}{v(x,y,z)} \qquad 1.3$$

where s is distance measured along the ray; M and M_o are points on the ray; v is the local wave velocity given by

$$v_p = \left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}}$$

for P or compressional waves and

$$v_s = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$$

for SH and SV or shear waves. The ray, everywhere parallel to the gradient of the phase function τ, is obtained by solving the Euler equations for the extremals of the integral in 1.3.

The amplitude coefficients are governed by the

transport equations (see e.g. Cerveny and Ravindra 1971 eq. 2.25' and 2.37) which can be integrated to give

$$U_i(M) = (-1)^m \left[\frac{(v\rho d\sigma)_{M_o}}{(v\rho d\sigma)_M} \right]^{\frac{1}{2}} U_i(M_o); \; i=1,2,3 \qquad 1.4$$

Here the index m is zero for all P and SH rays. For SV rays it assumes the value one if the ray has a turning point between M_o and M. Otherwise m equals zero.

The amplitude coefficients 1.4 have a very simple physical interpretation. The magnitude of U_i varies precisely in accordance with the principle of conservation of energy flux within a narrow tube of rays where the energy propagates only along the rays and no energy flows through the side walls of the ray tube. Thus each ray is a distinctive path of energy transport between the source and receiver.

The considerations so far require that the elastic parameters of the medium be continuous along the ray. The results can be easily generalized to include reflections and transmissions at interfaces. It is assumed that the medium consists of layers in welded contact so that displacement and traction are continuous across the interfaces. From the six equations expressing this continuity, it follows that in the zeroth approximation of ART (see e.g. Cerveny and Ravindra 1971, section 2.3)

1. SH waves propagate in complete independence of the other waves.
2. conversion between P and SV waves can occur by reflection and transmission at interfaces.
3. energy partitioning at interfaces can be described by plane wave reflection and transmission coefficients.

Using 1.4 for the continuation of the displacement vector along the ray within a continuous layer and using plane wave reflection and transmission coefficients for continuation across interfaces, it is easy to construct the displacement vector for a ray that has encountered an arbitrary number of interfaces between the points M_o and M (see Fig. 1). If we write

$$\vec{u}(M) = S(\omega)U(M)\, e^{-i\omega\tau}\vec{n}(M) \qquad 1.5$$

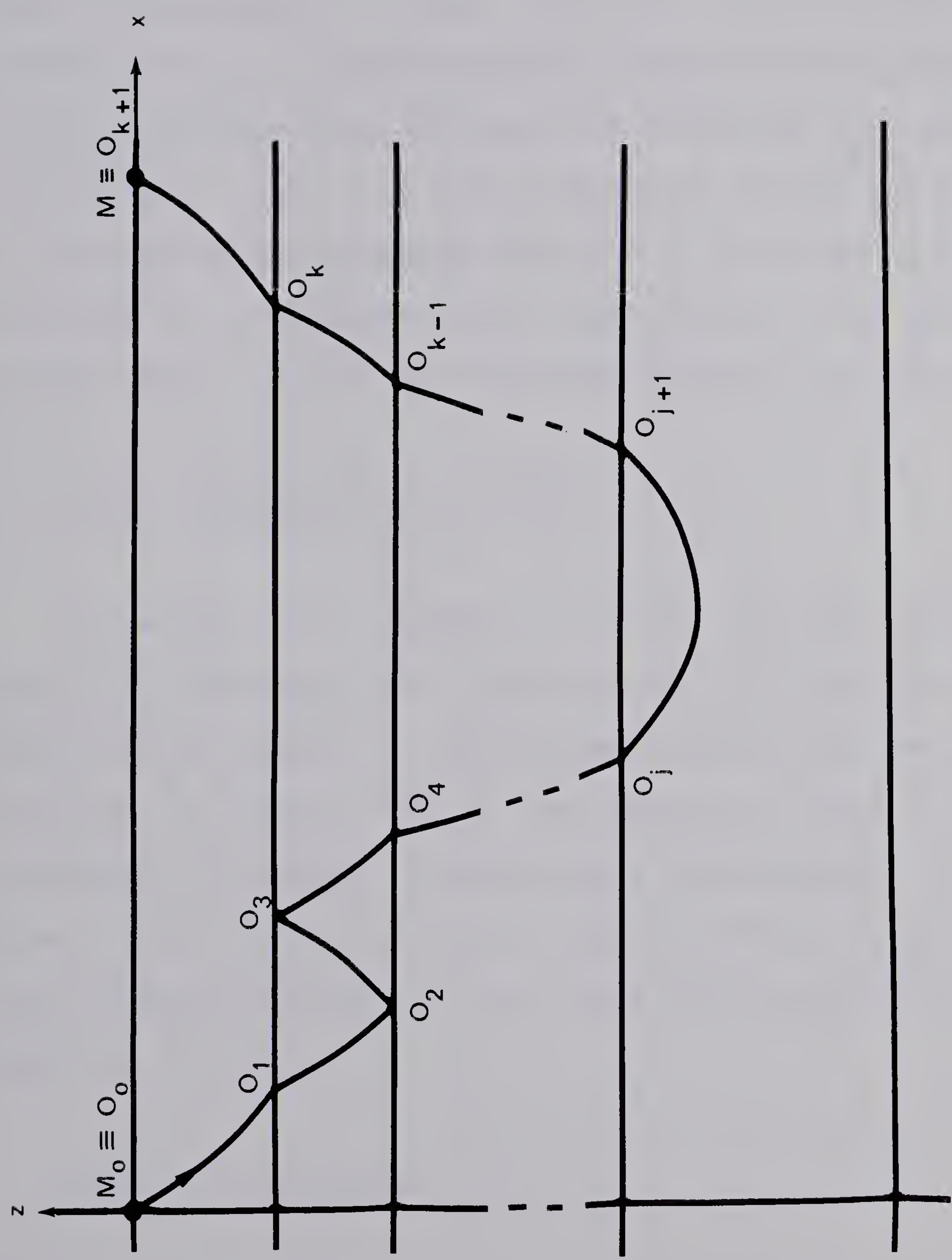
where $\vec{n}(M)$ may be $\vec{n}_p(M)$, $\vec{n}_{SV}(M)$, or $\vec{n}_{SH}(M)$ depending on the type of wave observed at M, then

$$\tau(M) = \tau(M_o) + \sum_{j=1}^{k+1} \int_{O_{j-1}}^{O_j} \frac{ds}{v} \; ; \; O_o \equiv M_o,\ O_{k+1} \equiv M \qquad 1.6$$

and

$$U(M) = \frac{(-1)^{\varepsilon}}{L(M)} \left[\frac{(v\rho)_{M_o}}{(v\rho)_M}\right]^{\frac{1}{2}} \prod_{j=1}^{k} \left[\frac{(v\rho)_{O_j^-}}{(v\rho)_{O_j^+}}\right]^{\frac{1}{2}} R(O_j) \qquad 1.7$$

Figure 1 Typical ray path in a layered, vertically inhomogeneous medium. M_o and M are respectively the source and receiver points; O_j , $j=1,\ldots,k$ are the points of incidence at interfaces.


x
z
M ≡ O_{k+1}
M_o ≡ O_o
O_1
O_2
O_3
O_4
O_j
O_{j+1}
O_{k-1}
O_k

where, for simplicity, $U(M_o)$ has been set to unity (only relative amplitudes are significant in synthetic seismogram computations); ε is the number of turning SV ray segments; $R(O_j)$ is the appropriate reflection and transmission coefficient at the point of incidence O_j; the superscripts - and + indicate properties at the point O_j on the side of the incident wave and on the side of the reflected or transmitted waves, respectively; and the function $L(M)$, called the spreading function, is given by

$$L(M) = \left[\frac{d\sigma(M)}{d\sigma(M_o)}\right]^{\frac{1}{2}} \prod_{j=1}^{k} \left[\frac{d\sigma(O_j^+)}{d\sigma(O_j^-)}\right]^{\frac{1}{2}}$$

In general, the spreading function may be evaluated using 1.2. However, for a point source in a vertically inhomogeneous medium, a rather simple expression may be obtained. We assume that in the immediate vicinity of the source the medium is essentially homogeneous. Let M_o be a point at unit distance from the source within this homogeneous region. Then since the wavefront is spherical at M_o:

$$d\sigma(M_o) = \sin\theta_o d\theta_o d\phi_o$$

where θ_o and ϕ_o are spherical polar coordinates describing the ray at M_o. From simple geometrical considerations the ray tube cross-sectional area at M can be

written as (see Fig. 2)

$$d\sigma(M) = r\frac{\partial r}{\partial \theta_o} \cos\theta(M)\, d\phi_o d\theta_o \ . \qquad 1.8$$

Furthermore, at the point O_j at the end of the j-th ray segment we have (see Fig. 2)

$$\left[\frac{d\sigma(O_j^+)}{d\sigma(O_j^-)}\right]^{\frac{1}{2}} = \left[\frac{\cos\theta(O_j^+)}{\cos\theta(O_j^-)}\right]^{\frac{1}{2}}$$

so that the spreading function can be written as

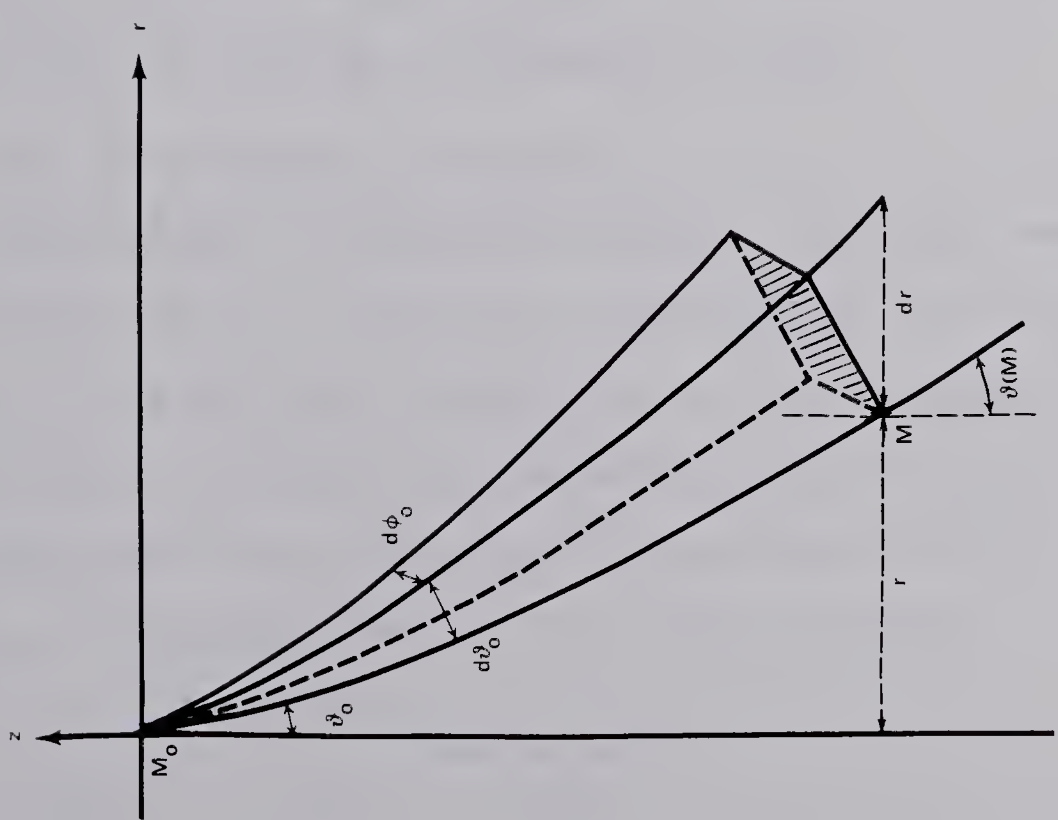
$$L(M) = \left[r\frac{\partial r}{\partial \theta_o} \frac{\cos\theta(M)}{\sin\theta_o}\right]^{\frac{1}{2}} \prod_{j=1}^{k} \left[\frac{\cos\theta(O_j^+)}{\cos\theta(O_j^-)}\right]^{\frac{1}{2}} \qquad 1.9$$

Equations 1.5 to 1.7 and 1.9 constitute the zeroth approximation of the particle displacement provided by ART in its basic form. Its deficiency is apparent as an infinite amplitude is predicted at points where the ray tube cross-sectional area $d\sigma$ vanishes. Since $d\sigma$ is also the Jacobian of transformation between spatial and ray coordinates, it is clear that the above results are invalid not only at these points but also in regions along the ray beyond these points (Babich 1961). The following chapters aim to resolve these difficulties.

Figure 2 Ray tube cross-sectional area.

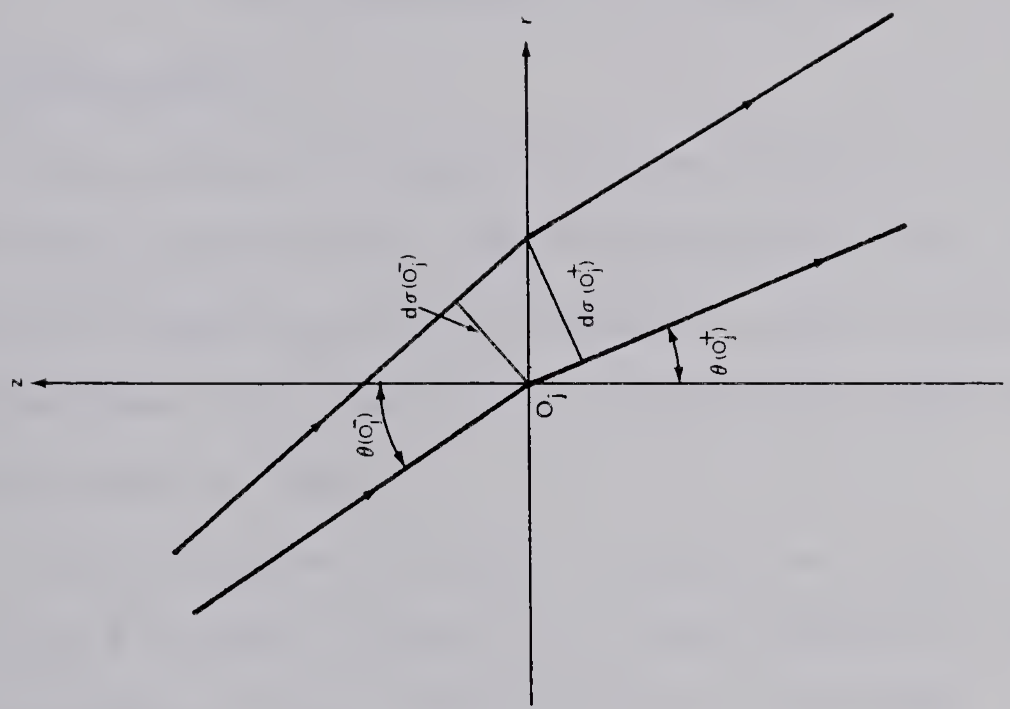
(a) Cross-section at the point M of an elementary ray tube originating at the point M.

(b) Change of the cross-sectional area across an interface.



z
r
M_o
ϑ_o
$d\vartheta_o$
$d\Phi_o$
M
r
dr
$\vartheta(M)$

(a)



z
r
O_j
$\theta(O_j^-)$
$d\sigma(O_j^-)$
$d\sigma(O_j^+)$
$\theta(O_j^+)$

(b)

CHAPTER 2

RAY AMPLITUDE BY WAVE METHOD

A. Introduction

In the previous chapter the propagation of small amplitude elastic waves was studied using ART. The theory is useful but unfortunately, under certain circumstances the results are invalid and more exact methods must be used.

In this chapter a formal integral solution to the problem of a point source in a plane-layered, vertically inhomogeneous elastic medium will be obtained. In the following chapters, this integral solution will be evaluated to provide an extension to ART.

B. The Elastodynamic Equations

We consider a perfectly elastic, isotropic medium in which the wave velocities and density are functions only of the vertical coordinate z. For symmetry in the solution, we consider an explosive point source located at the point with cylindrical coordinate $(r,z,\phi) = (0,z_o,0)$ and described by the source function

$$P(r,z,t) = P_o(t)\,\frac{\delta(z-z_o)\,\delta(r)}{2\pi r} \quad .$$

Since the ray path in a vertically inhomogeneous medium is a plane curve, the azimuthal angle ϕ may be ignored. Within the medium, the equations of motion are

$$\nabla \cdot \overline{\overline{\sigma}} = \rho \frac{\partial^2 \vec{u}}{\partial t^2} - \vec{f} \qquad 2.1$$

where the body force per unit volume $\vec{f}$ is derivable from the source function by

$$\vec{f}(r,z,t) = -\nabla P(r,z,t)$$

and the stress dyadic $\overline{\overline{\sigma}}$ is related to the particle displacement $\vec{u}$ via the constitutive equations

$$\overline{\overline{\sigma}} = (\lambda \nabla \cdot \vec{u}) \overline{\overline{I}} + \mu \left[\nabla \vec{u} + (\vec{u} \nabla)^T \right]$$

$\overline{\overline{I}}$ is the unit dyadic and the superscript T indicates a transpose.

The partial differential equations 2.1 may be considerably simplified if transformed coordinates are used. We will use the Fourier transform in time defined by

$$\vec{u}(r,z,\omega) = \int_{-\infty}^{\infty} \vec{u}(r,z,t)\, e^{-i\omega t} dt \qquad 2.2$$

$$\vec{u}(r,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{u}(r,z,\omega)\, e^{+i\omega t} d\omega$$

and the zeroth-order Hankel transform in the cylindrical radius r defined by

$$\vec{u}(p,z,\omega) = \int_0^\infty \vec{u}(r,z,\omega)\, rJ_o(\omega pr)\, dr$$

$$\vec{u}(r,z,\omega) = \omega^2 \int_0^\infty \vec{u}(p,z,\omega)\, pJ_o(\omega pr)\, dp$$

For simplicity of notation, a function is distinguished from its transform only by its arguments.

With these transformations, the equations of motion and constitutive equations are reduced to a system of first order ordinary differential equations

$$\frac{d\vec{V}}{dz} = i\omega\overline{\overline{A}}\vec{V} + \vec{w} \qquad 2.3$$

where

$$\vec{V}^T(p,z,\omega) = \left[\frac{-1}{pr}\frac{\partial}{\partial r}(ru_r),\ \sigma_{zz},\ i\omega u_z,\ \frac{-1}{i\omega pr}\frac{\partial}{\partial r}(r\sigma_{rz})\right] \qquad 2.4$$

are the doubly transformed components of displacement and stress;

$$\vec{w}^T(p,t,\omega) = \left[0,\ -f_z,\ 0,\ \frac{1}{i\omega pr}\frac{\partial}{\partial r}(rf_r)\right] \qquad 2.5$$

are the doubly transformed force functions; and the matrix $\overline{\overline{A}}$ is given by

$$\bar{\bar{A}} = \begin{bmatrix} 0 & 0 & p & \frac{1}{\mu} \\ 0 & 0 & \rho & p \\ \frac{p\lambda}{\lambda+2\mu} & \frac{1}{\lambda+2\mu} & 0 & 0 \\ \rho - \frac{4p^2\mu(\lambda+\mu)}{\lambda+2\mu} & \frac{p\lambda}{\lambda+2\mu} & 0 & 0 \end{bmatrix}$$

The differential system 2.3 is well known in seismological literature and arises also in the study of SH and acoustical waves. After Gilbert and Backus (1966), we will call the non-singular matrix $\bar{\bar{V}}_f$ a fundamental matrix of this differential system if it is a solution of the homogeneous form of the system, i.e. if each of its columns satisfies

$$\frac{d\vec{V}}{dz} = i\omega\bar{\bar{A}}\vec{V} \qquad 2.6$$

If a fundamental matrix has the property that $\bar{\bar{V}}_f(z_o)$ is the identity matrix then it is called a propagator from z_o and is denoted by $\bar{\bar{P}}(z;z_o)$. Propagators can be constructed from any fundamental matrix by

$$\bar{\bar{P}}(z;z_o) = \bar{\bar{V}}_f(z)\,\bar{\bar{V}}_f^{-1}(z_o). \qquad 2.7$$

Using propagators, the solution of the homogeneous system 2.6 can be easily expressed as

$$\vec{V}(p,z,\omega) = \overline{\overline{P}}(z;z_o)\ \vec{V}(p,z_o,\omega) \tag{2.8}$$

while that for the inhomogeneous system 2.3 is

$$\vec{V}(p,z,\omega) = \overline{\overline{P}}(z;z_o)\ \vec{V}(p,z_o,\omega) + \int_{z_o}^{z} \overline{\overline{P}}(z;\xi)\vec{w}(p,\xi,\omega)\,d\xi\ . \tag{2.9}$$

Thus we see that the basic problem in solving the elastodynamic equations is to find a fundamental matrix $\overline{\overline{V}}_f$.

C. The Fundamental Matrix

The matrix $\overline{\overline{A}}$ has eigenvalues $\pm q_p$ and $\pm q_s$ where

$$q_p = \left(\frac{1}{v_p^2} - p^2\right)^{\frac{1}{2}} \ ; \quad q_s = \left(\frac{1}{v_s^2} - p^2\right)^{\frac{1}{2}}$$

v_p and v_s being the local P and S wave velocities. The corresponding eigenvectors, normalized with respect to energy flux in the z-direction, are

$$\vec{V}^T_{\pm p} = \frac{1}{(2\rho q_p)^{\frac{1}{2}}}\ (-p, \mu\Gamma, \mp q_p, \pm 2\mu p q_p)$$

$$\vec{V}^T_{\pm s} = \frac{i}{(2\rho q_s)^{\frac{1}{2}}}\ (\pm q_s, \pm 2\mu p q_s, -p, -\mu\Gamma)$$

where

$$\Gamma = q_s^2 - p^2 \quad ; \quad i = (-1)^{\frac{1}{2}}$$

If we define a matrix $\overline{\overline{N}}$ using the eigenvectors as columns

$$\overline{\overline{N}} = \left\{ \vec{V}_{+p}, \vec{V}_{-p}, \vec{V}_{+s}, \vec{V}_{-s} \right\}$$

then, using $\overline{\overline{N}}$, we can diagonalize the matrix $\overline{\overline{A}}$ by a similarity transformation yielding the matrix of eigenvalues

$$\overline{\overline{Q}} = \overline{\overline{N}}^{-1} \, \overline{\overline{A}} \, \overline{\overline{N}} = \begin{bmatrix} -q_p & 0 & 0 & 0 \\ 0 & q_p & 0 & 0 \\ 0 & 0 & -q_s & 0 \\ 0 & 0 & 0 & q_s \end{bmatrix}$$

Thus it follows that

$$\bar{\bar{N}}^{-1}\,\bar{\bar{A}} = \bar{\bar{Q}}\,\bar{\bar{N}}^{-1}$$

Using this relationship the homogeneous system 2.6 can be written as

$$\frac{d}{dz}\,(\bar{\bar{N}}^{-1}\,\vec{V}) = i\omega\bar{\bar{Q}}(\bar{\bar{N}}^{-1}\,\vec{V}) + \bar{\bar{B}}(\bar{\bar{N}}^{-1}\,\vec{V})$$

where

$$\bar{\bar{B}} = \frac{d\bar{\bar{N}}^{-1}}{dz}\,\bar{\bar{N}}$$

Notice that $\bar{\bar{B}}$ involves the derivatives of the elastic parameters as well as terms in inverse powers of q_p and q_s ; also it is one order in frequence lower than $i\omega\bar{\bar{Q}}$. Therefore, if the problem is such that the WKBJ conditions are satisfied, i.e., if the variations in the elastic parameters of the medium are small over a wavelength and that the solution is not required near turning points where either q_p or q_s vanishes, then we can ignore the term in $\bar{\bar{B}}$ and approximate the homogeneous system 2.6 by

$$\frac{d}{dz}\,(\bar{\bar{N}}^{-1}\,\vec{V}) = i\omega\bar{\bar{Q}}(\bar{\bar{N}}^{-1}\,\vec{V})$$

for which $e^{i\omega\int_{-\infty}^{z}\bar{\bar{Q}}(\xi)\,d\xi}$ is a solution matrix.

In the WKBJ approximation, therefore, a fundamental matrix of the elastodynamic equations 2.3 is

$$\bar{\bar{V}}_f = \bar{\bar{N}} e^{i\omega \int_{-\infty}^{z} \bar{\bar{Q}}(\xi) d\xi} . \qquad 2.10$$

D. The Ray Expansion

Cisternas et al (1973) and Kennett (1974) have shown that for a stack of plane-parallel, homogeneous layers the solution to the elastodynamic equations can be decomposed into an infinite number of individual contributions. Identical with the rays in ART, each of these contributions is independently a solution of 2.3 and characterized by a distinctive path of energy transport. For media consisting of vertically inhomogeneous layers where the WKBJ approximation is valid, the ray expansion is again valid (Kennett 1974).

In the following, the complete wave solution to 2.3 for a layered, inhomogeneous medium need not be attempted. For comparison with ART results, it suffices to consider the contribution of an arbitrary ray in the ray expansion.

E. The Source Wave

We will denote as the source region a volume about the source the linear dimension of which is small enough so that the elastic parameters within it can be regarded as constant but large enough compared with the wavelength.

For weakly inhomogeneous media where the WKBJ conditions are satisfied such a region can always be defined.

By a source wave we will mean a wave propagating within the source region from the source without experiencing any reflection or transmission at interfaces.

We assume that both the source and receiver are located on a free surface at $z = z_o$ so that waves exist only for $z<z_o$. Thus the problem for a source wave is equivalent to that of wave propagation within a homogeneous half space for which 2.9 provides the complete solution. Using 2.5, 2.7, 2.10 and the radiation condition this solution is

$$\vec{V}^T(p,M_s,\omega) = \frac{i\omega P_o(\omega) e^{-i\omega\left|\int_{z_o}^{z_s} q_p(\xi)\,d\xi\right|}}{\pi v_p^2(z_o)(2\rho q_p)_{M_s}^{\frac{1}{2}}(2\rho q_p)_{M_o}^{\frac{1}{2}}} \qquad 2.11$$

$$\cdot\ (p,\ -\mu\Gamma,\ -q_p,\ 2\mu p q_p)_{M_s};\ z_o>z_s$$

where $M_s=(r_s,z_s)$ denotes a point within the source region.

For the computation of synthetic seismograms the particle displacement $\vec{u}$ is desired. This can be obtained from the verticle displacement u_z by

$$\vec{u} = \frac{u_z}{(\vec{n}\cdot\vec{n}_z)}\ \vec{n}$$

where $\vec{n}_z$ is the unit vector in the positive z-direction

and $\vec{n}$ may be $\vec{n}_p$ or $\vec{n}_{SV}$ for P and SV waves respectively. Using this relation and the definition 2.4 of $\vec{V}$, the source wave displacement at M_s is

$$\vec{u}(p,M_s,\omega) = \frac{S(\omega)}{i\omega q_p(M_o)} \frac{\left[(\rho/q_p)^{\frac{1}{2}}\cos\theta\right]_{M_o}}{\left[(\rho/q_p)^{\frac{1}{2}}\cos\theta\right]_{M}} e^{-i\omega\left|\int_{z_o}^{z_s} q_p(\xi)\,d\xi\right|} \vec{n}_p \qquad 2.12$$

where θ is the acute angle between $\vec{n}_p$ and $\vec{n}_z$ and

$$S(\omega) = \left[\frac{i\omega P_o(\omega) q_p}{2\pi\rho v_p^2 \cos\theta}\right]_{M_o}$$

is, apart from a multiplicative constant, the spectrum of the source pulse.

F. Propagation Along a Ray

Outside the source region wave propagation is described by the homogeneous system 2.6. Independent solutions are columns of the fundamental matrix $\overline{\overline{V}}_f$:

$$\vec{V}_{p\pm} = C_{\pm p}\, \vec{V}_{\pm p}\, e^{i\omega \int_{-\infty}^{z} \pm q_p(\xi)\,d\xi}$$

representing upgoing and downgoing P waves and

$$\vec{V}_{s\pm} = C_{\pm s}\, \vec{V}_{\pm s}\, e^{i\omega \int_{-\infty}^{z} \pm q_s(\xi)\,d\xi}$$

representing upgoing and downgoing SV waves. $C_{\pm p}$ and

$C_{\pm s}$ are arbitrary constants.

At any two points $M_1=(r_1,z_1)$ and $M_2=(r_2,z_2)$ within a continuous volume outside the source region, the wave solutions are related through the propagator by 2.8, i.e.

$$\vec{V}(p,M_2,\omega) = \overline{\overline{N}}(M_2)\, e^{i\omega\int_{z_1}^{z_2} \overline{\overline{Q}}(\xi)\,d\xi}\, \overline{\overline{N}}^{-1}(M_1)\vec{V}(p,M_1,\omega) \qquad 2.13$$

Assuming turning points do not exist along the ray so that the WKBJ approximation is valid throughout the region under consideration, 2.13 together with 2.11 can be used to show that if the wave at M_1 is of a given type, a downgoing P wave say, then it will also be of the same type at M_2. In other words, reflections off the velocity gradient, and their associated mode conversions, are negligible in the WKBJ approximation. The complex amplitudes at the two points are related by

$$u(p,M_2,\omega) = \frac{G(M_1)}{G(M_2)}\, e^{-i\omega\left|\int_{z_1}^{z_2} q(\xi)\,d\xi\right|}\, u(p,M_1,\omega) \qquad 2.14$$

where, for P waves

$$q = q_p \;;\quad G = \left(\frac{\rho}{q_p}\right)^{\frac{1}{2}}\cos\theta \qquad 2.15$$

and for SV waves

$$q = q_s \;;\quad G = (\rho q_s)^{\frac{1}{2}}\,\frac{\sin\theta}{p} \qquad 2.16$$

and

$$\vec{u}(p,M,\omega) = u(p,M,\omega)\vec{n}$$

If a turning point exists along the ray between M_1 and M_2, the WKBJ result 2.14 is no longer valid. However, only a slight modification is needed to provide the correct expression. Appendix 1 shows that the effect of a turning point can be represented by the reflection coefficient $e^{i\frac{\pi}{2}}$. Suppose a turning point exists at (r^*,z^*), then the complex amplitudes at M_1 and M_2 are related by

$$u(p,M_2,\omega) = (-1)^m \frac{G(M_1)}{G(M_2)} e^{i\frac{\pi}{2} - i\omega\left\{\left|\int_{z^*}^{z_1} q(\xi)d\xi\right| + \left|\int_{z^*}^{z_2} q(\xi)d\xi\right|\right\}} \cdot u(p,M_1,\omega). \qquad 2.17$$

The $(-1)^m$ factor is introduced for the same reason as in Chapter 1 for SV waves.

G. WKBJ Solution for an Arbitrary Ray

When a wave is incident upon an interface, it will either be reflected or transmitted and mode conversion can occur. The complex amplitude u_I of the incident wave and the complex amplitude u_R of the reflected or transmitted wave are related through the appropriate plane wave reflection and transmission coefficients R

(as given by Cerveny and Ravindra 1971, section 2.3):

$$u_R = Ru_I \qquad 2.18$$

Thus, starting with the source wave displacement 2.12, the relations 2.14, 2.17 and 2.18 can be used to construct the particle displacement of an arbitrary ray which has gone through n turning points and suffered k reflections and transmissions at the points O_j, $j = 1,2,\ldots,k$, before reaching receiver point M. The particle displacement at M is

$$u(p,M,\omega) = \frac{S(\omega)(-1)^{\varepsilon}G(M_o)}{i\omega q_p(M_o)G(M)} e^{-i\omega\psi(p)+in\frac{\pi}{2}} \prod_{j=1}^{k} \frac{G(O_j^-)}{G(O_j^+)} R(O_j) \qquad 2.19$$

where

$$\psi(p) = \sum_{j=1}^{k+1} \psi_j(p)$$

with

$$\psi_j(p) = \left| \int_{z_{j-1}}^{z_j} q(p,\xi)\,d\xi \right|$$

if the ray has no turning point between $O_{j-1}=(r_{j-1},z_{j-1})$ and $O_j=(r_j,z_j)$, and

$$\psi_j(p) = 2\left| \int_{z_j^*}^{z_j} q(p,\xi)\,d\xi \right|$$

if the ray has a turning point at (r_j^*, z_j^*) between O_{j-1} and O_j.

Inverting the Hankel transform and keeping only the first term of the asymptotic expansion for the Hankel function (Abramowitz and Stegun 1965), we obtain from 2.19 the solution in the frequency domain

$$u(M,\omega) = (-1)^{\varepsilon} S(\omega) \left[\frac{\omega}{2\pi r}\right]^{\frac{1}{2}} e^{i(n\frac{\pi}{2} - \frac{\pi}{4})} \int_{-\infty}^{\infty} f(p)\, e^{-i\omega\tau(p,r)} dp \qquad 2.20$$

where

$$f(p) = \frac{p^{\frac{1}{2}} G(M_o)}{q_p(M_o) G(M)} \prod_{j=1}^{k} \frac{G(O_j^-)}{G(O_j^+)} R(O_j) \qquad 2.21$$

and

$$\tau(p,r) = pr + \psi(p) \qquad 2.22$$

is the phase function. This is the plane wave approximation of the WKBJ solution for the displacement amplitude of an arbitrary ray.

CHAPTER 3

PHASE SHIFT DUE TO CAUSTICS

A. Introduction

A caustic may be defined as a point at which $d\sigma$, the cross-sectional area of the ray tube vanishes. The basic theoretical problems connected with caustics have been investigated by Brekhovskih (1960), Ludwig (1966), Sato (1969) and others.

Essentially, in the high frequency approximation, the dynamic properties of the wave field are characterized by a $\pi/2$ phase shift of the harmonic components at the caustic and a concentration of wave energy in its immediate neighborhood. For pulse waveforms, this $\pi/2$ phase shift results in a marked distortion of the pulse shape, providing a useful criterion for proper identification of later phases on a seismogram (Choy and Richards 1975). Similarly, the larger amplitude near a caustic renders it important for the interpretation of seismograms.

In this chapter, the phase shift at a caustic will be established. This will lead to a useful expression relating the phase shift to the number of turning points of the ray. Ray amplitudes near a caustic will be dealt with in the next chapter.

B. Relation between Turning Points and Caustics

The occurrence of caustics and their associated

phase shifts is closely related to the occurrence of turning points of totally reflected waves. In fact, the pulse distortion observed in totally reflected waves had often been erroneously ascribed to turning points. The origin of this mistaken association is probably due to the fact that turning points coincide with caustics in the limiting case of totally reflected plane waves (Tolstoy 1968, Silbiger 1968).

In general, for waves generated by a point source, caustics are distinct from turning points. Indeed, a pulse may travel through a turning point and yet not form a caustic and therefore not experience any distortion. Direct P waves in the Earth are obvious examples of this situation.

A caustic corresponds to the vanishing of the ray tube cross-sectional area. For vertically inhomogeneous media, equation 1.8 implies that $\partial r/\partial p$ vanishes at a caustic, where $p = \sin\theta/v$ is the geometrical ray parameter. Thus the existence of caustics can be easily established by examining the variations in the sign of the derivative $\partial r/\partial p$ along the ray path.

For this purpose, it is convenient to divide the total ray path into segments containing single turning points (called here turning segments), and segments with no turning points (called here non-turning segments). We will denote by c_i the ray segment bounded by s_{i-1} and s_i, where s is the distance along the ray measured from the

source point. Then, the value of $\left(\frac{\partial r}{\partial p}\right)_s$ on the ray at s is given by the sum of contributions from individual segments:

$$\left(\frac{\partial r}{\partial p}\right)_s = \sum_{i=1}^{N} \left(\frac{\partial r}{\partial p}\right)_{c_i}$$

For non-turning segments, its contribution is given by the standard integral

$$\left(\frac{\partial r}{\partial p}\right)_{c_i} = \int_{\min(z_{i-1},z_i)}^{\max(z_{i-1},z_i)} \frac{d\xi}{v^2 q^3} > 0 \qquad 3.1$$

where z_{i-1} and z_i are the z-coordinates of the points denoted by s_{i-1} and s_i.

For turning segments, more care is required. We assume that the velocity function near a turning point can be expanded in a Taylor series in powers of $z-z^*$, z^* being the z-coordinate of the turning point. We can always define the turning segments short enough that only the linear term in the expansion need be considered in approximating the velocity function over the entire segment; i.e.

$$v(z) = v(z^*) - g(z-z^*) \qquad 3.2$$

where

$$g = \left| \frac{dv(z^*)}{dz} \right|$$

is assumed to be non-zero. This assumption is invalid only in the special cases where the turning point coincides with an extremum of the velocity function and higher order terms must be considered. Using 3.2, the contribution of a turning segment to $\frac{\partial r}{\partial p}$ can easily be found to be

$$\left(\frac{\partial r}{\partial p}\right)_{c_i} = \frac{-1}{gp^2}\left[\left(\frac{1}{vq}\right)_{z_{i-1}} + \left(\frac{1}{vq}\right)_{z_i}\right] < 0 \qquad 3.3$$

We now consider a ray with n turning points at s_i^*, $i=1,2,...,n$. The number of caustics associated with this ray can be found by simply counting the number of sign changes in $(\frac{\partial r}{\partial p})_s$ as s varies from 0 to s_M, s_M being the distance of the observation point along the ray.

Between the source and the first turning point $(\frac{\partial r}{\partial p})_s$ has contributions only from non-turning segments and therefore is always positive. As s approaches s_1^*, $(\frac{\partial r}{\partial p})_s$ tends to positive infinity since the integrand in 3.1 has a non-integrable singularity at the turning point where q vanishes.

At the turning point $(\frac{\partial r}{\partial p})_s$ is undefined. Physically, this is expected since a ray with a slightly larger ray parameter than a reference ray can not possibly penetrate to the same depth as the turning point of the reference ray. In this case, the cross-sectional area of the ray tube can be expressed alternatively by

$$d\sigma = r \left|\frac{\partial z}{\partial \theta_o}\right| d\theta_o \, d\phi_o$$

Since $\partial z/\partial \theta_o$ is obviously positive at a turning point, $d\sigma$ is finite and caustics never coincide with turning points for waves from point sources. Therefore there can be no caustics before the first turning point.

Just past the turning point, for $s = s_1^* + \delta$ where δ is positive and infinitesimally small, $(\frac{\partial r}{\partial p})_s$ jumps discontinuously to negative infinity by 3.3. In general

$$\left(\frac{\partial r}{\partial p}\right)_s \rightarrow \pm \infty \text{ as } s \rightarrow s_i^* \quad ; \quad s \lessgtr s_i^* \qquad 3.4$$

Between consecutive turning points s_i^* and s_{i+1}^*, additional contributions again come only from non-turning segments. Therefore $(\frac{\partial r}{\partial p})_s$ increases monotonically from negative to positive infinity vanishing exactly once for some s_c where $s_i^* < s_c < s_{i+1}^*$. In other words, exactly one caustic must exist between two consecutive turning points.

Between the last turning point s_n^* and the observation point s_M an additional caustic may or may not exist. Using 3.4 and keeping in mind that $(\frac{\partial r}{\partial p})_s$ increases monotonically from $s_n^* + \delta$ to s_M, we conclude that an additional caustic exists if and only if $(\frac{\partial r}{\partial p})_{s_M} > 0$.

The above considerations are summarized schematically in Fig. 3 where $(\frac{\partial r}{\partial p})_s$ is plotted as a function of s. It is clear that for a ray with n turning points, the number of

caustics formed along the ray is given by

$$n_c = n - \tfrac{1}{2}\left[1-\mathrm{sgn}\left(\frac{\partial r}{\partial p}\right)\right]_M \qquad 3.5$$

Ray diagrams illustrating this relationship for layered media are shown in Fig. 4.

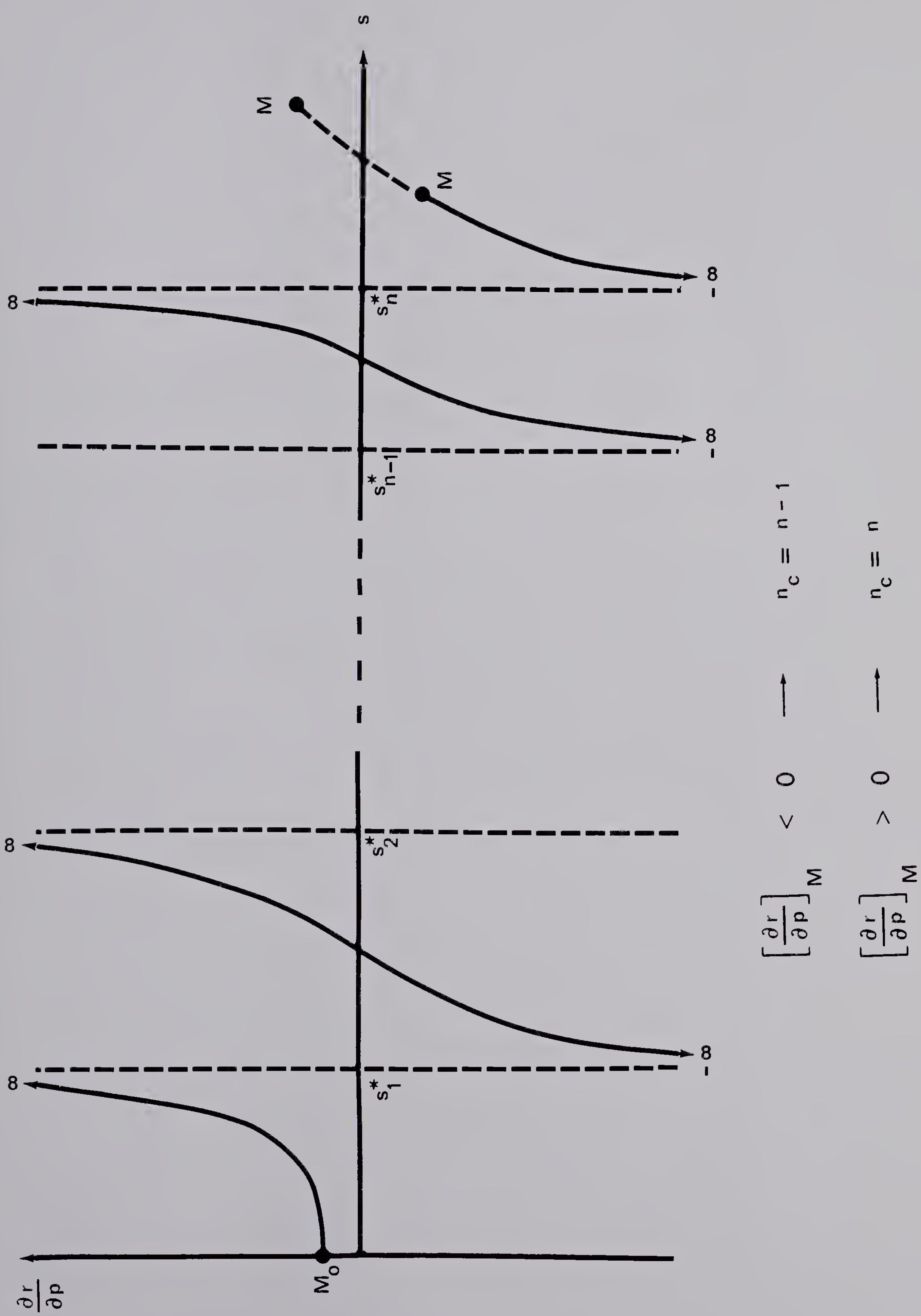
C. Phase Shift at a Caustic

It is well known that the high frequency harmonic components of a wave experience a discontinuous $\pi/2$ phase shift as the wave propagates through a caustic along a ray. This $\pi/2$ phase shift has been derived by Landau and Lifshitz (1951), Kay and Keller (1954), among others. In this section, an alternative derivation is presented. It not only produces the expected phase shift but also provides a rule for extending the validity of the asymptotic ray theory solution to regions beyond a caustic.

We assume that a caustic is formed at the point $M_c = (r_c, z_c)$. Analogous to the source region of the previous chapter, we define the caustic region as a neighborhood of M_c small enough so that it can be regarded as homogeneous but large compared with the wavelength. To study the effects of passage through a caustic, it suffices to consider wave propagation within the homogeneous caustic region.

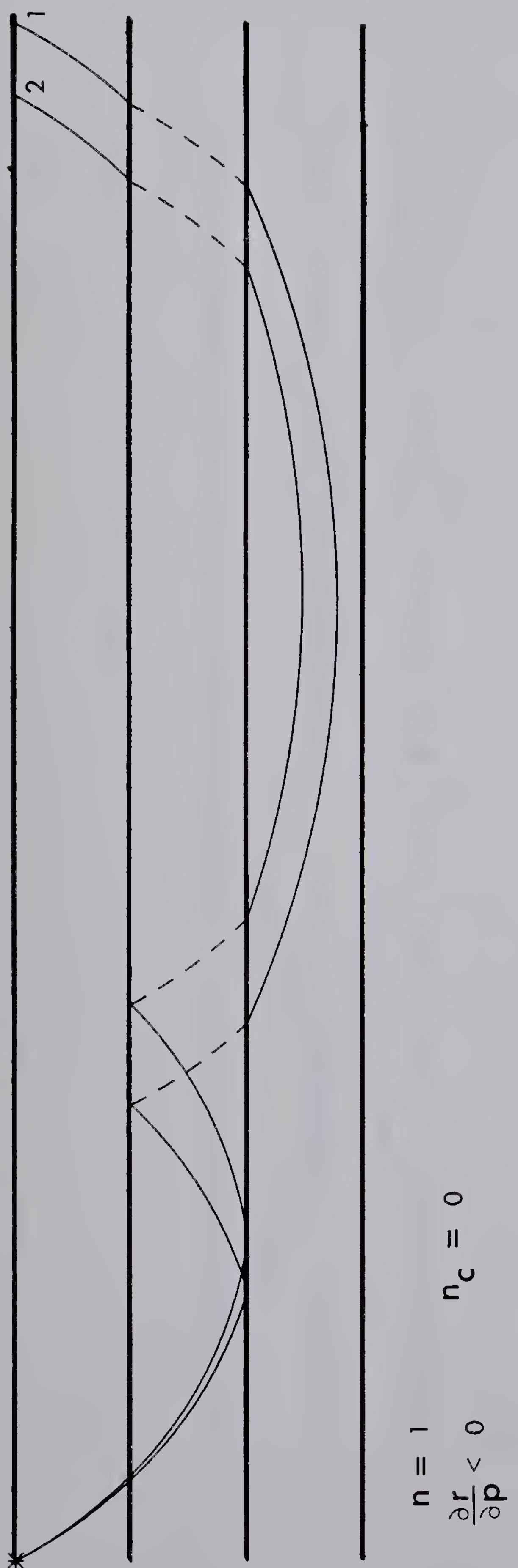
Inside the caustic region, independent solutions of the transformed wave equations 2.6 are $\exp(\pm i\omega qz)$. Therefore the general solution in the frequency domain is

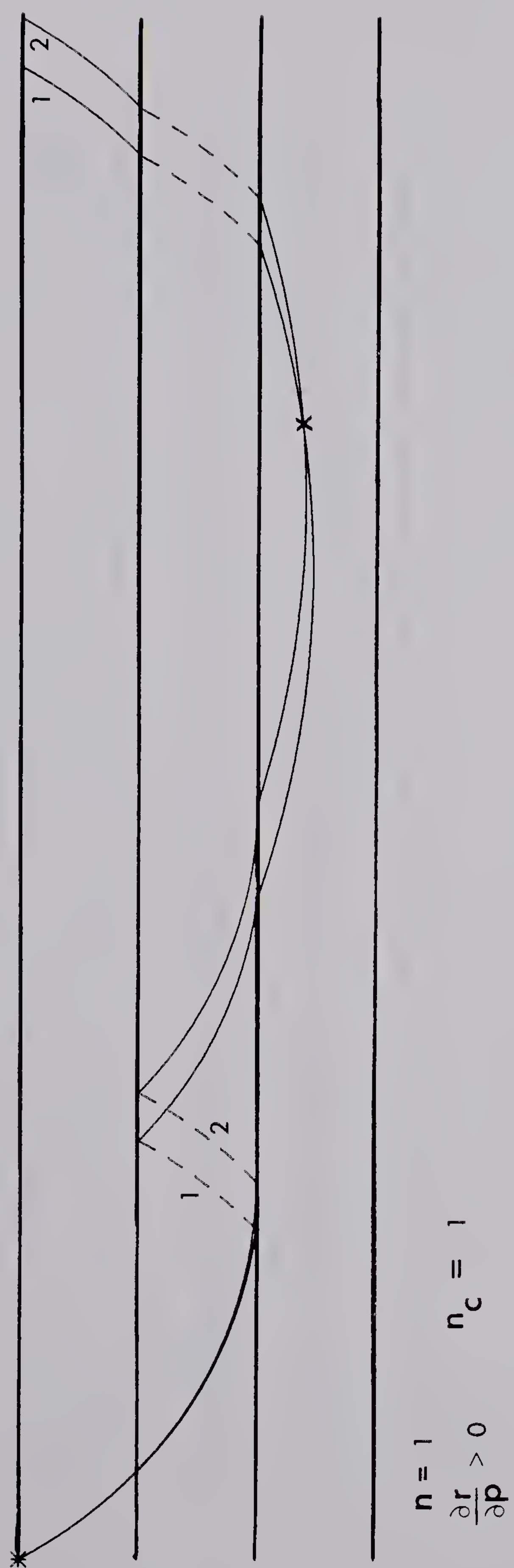
Figure 3 Variation of $\partial r/\partial p$ with arc length s along a typical ray path. Dependence of n_c on n and $(\partial r/\partial p)_M$ is apparent.

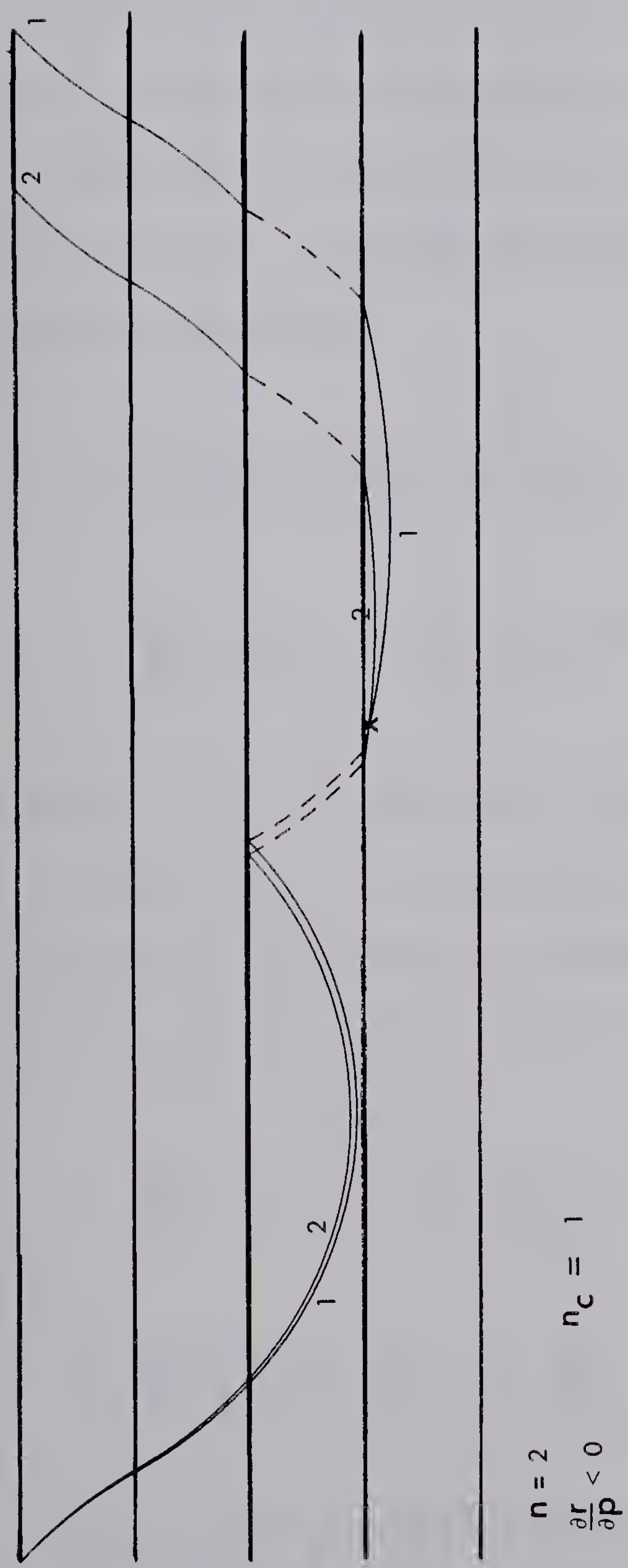


$\frac{\partial r}{\partial p}$
M_o
s
M
M
∞
$-\infty$
s_1^*
s_2^*
s_{n-1}^*
s_n^*
$\left[\frac{\partial r}{\partial p}\right]_M < 0 \longrightarrow n_c = n - 1$
$\left[\frac{\partial r}{\partial p}\right]_M > 0 \longrightarrow n_c = n$

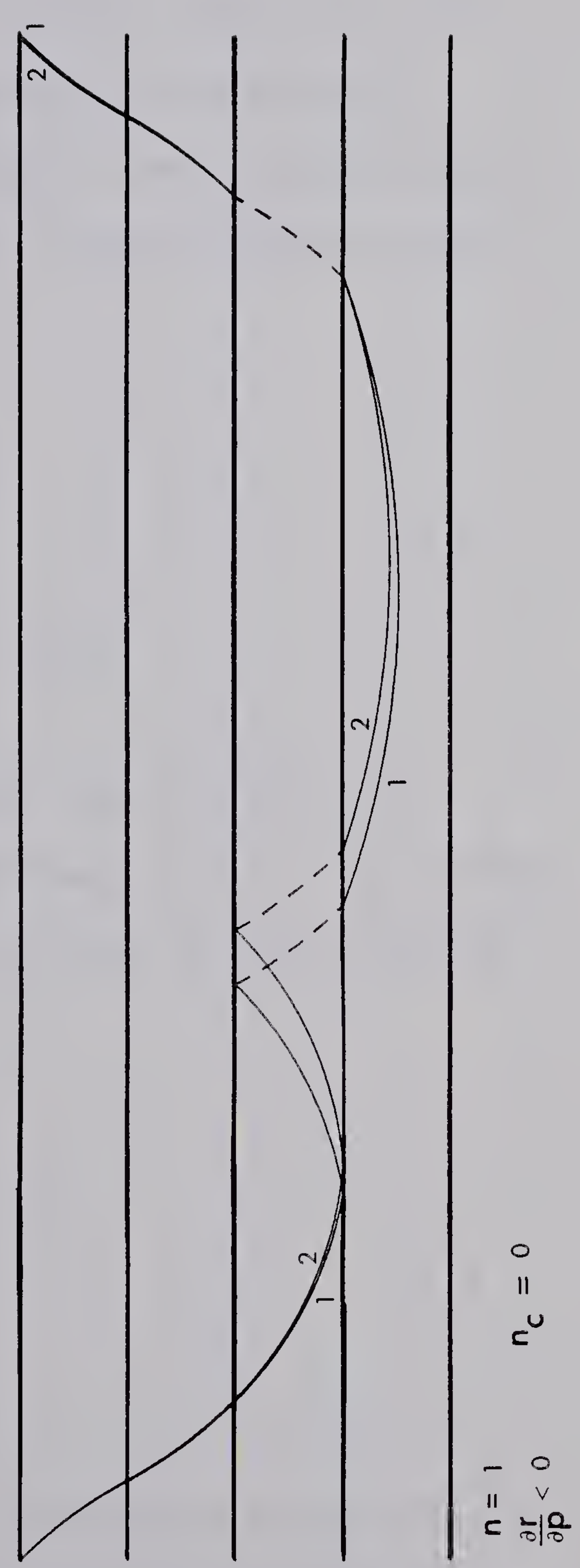
Figure 4 Examples of the application of equation 3.5. The ray with the smaller (bigger) ray parameter is labelled ray 1(2). Solid (dashed) curves represent P (SV) ray segments. Caustics are marked by X.


1
2
n = 1
$\frac{\partial r}{\partial p} < 0$
$n_c = 0$


2
1
1
2
n = 1
$\frac{\partial r}{\partial p} > 0$
$n_c = 1$



n = 2
$\frac{\partial r}{\partial p} < 0$
$n_c = 1$
1
2



n = 1
$\frac{\partial r}{\partial p} < 0$
$n_c = 0$
1
2

$$u(r,z,\omega) = \sum_{\pm} \int_{-\infty}^{\infty} \frac{a_{\pm}(p)}{(r)^{\frac{1}{2}}} e^{-i\omega(pr \pm qz)} dp \qquad 3.6$$

where the summation is over the $\pm$ indices; and $a_{\pm}(p)$ are functions to be determined. We prescribe the initial conditions at a point $M_a = (r_a, z_a)$ within the caustic region. In general, a harmonic component of a high frequency wave pulse satisfies

$$\begin{aligned} u(M_a) &= [A\, e^{-i\omega\tau}]_{M_a} \equiv g(M_a) \\ \frac{\partial u}{\partial z}(M_a) &= [-i\omega A \frac{\partial \tau}{\partial z} e^{-i\omega\tau}]_{M_a} \equiv h(M_a) \end{aligned} \qquad 3.7$$

where A is an amplitude function and only the term of highest order in frequency has been kept in 3.7. We assume that the direction of propagation is from M_a to M_c defined by the directional cosines

$$\begin{aligned} \frac{\partial \tau}{\partial r}(M_a) &= p_r \equiv \frac{\partial \tau}{\partial r_a} \\ \frac{\partial \tau}{\partial z}(M_a) &= \pm q_r \equiv \frac{\partial \tau}{\partial z_a} \end{aligned} \qquad 3.8$$

where p_r and q_r are the geometrical ray parameter and the vertical wave slowness and the $\pm$ signs hold for downgoing and upgoing waves respectively.

Application of the initial conditions 3.7 to the

general solution 3.6 yields, with the help of the Fourier integral theorem

$$a_{\pm}(p) = \frac{r_a^{\frac{1}{2}}\omega}{4\pi} \int_{-\infty}^{\infty} \left[g(\xi,z_a) \mp \frac{h(\xi,z_a)}{i\omega q} \right] e^{i\omega(p\xi \pm qz_a)} \, d\xi$$

Substituting this into 3.6 we obtain

$$u(r,z,\omega) = \frac{\omega}{4\pi}\left(\frac{r_a}{r}\right)^{\frac{1}{2}} \sum_{\pm} \iint_{-\infty}^{\infty} \left[1 \pm \frac{\partial\tau}{\partial z_a} / q \right] A(\xi,z_a)\, e^{-i\omega\Phi^{\pm}} \, d\xi dp \qquad 3.9$$

where

$$\Phi^{\pm} = \pm q(z-z_a) + p(r-\xi) + \tau(\xi,z_a) \qquad 3.10$$

are the phase functions.

The stationary points $W^{\pm} = (p_W, \xi_W)$ of the phase functions are defined by

$$\frac{\partial\Phi^{\pm}}{\partial p}(p_W,\xi_W) = 0 \quad ; \quad \frac{\partial\Phi^{\pm}}{\partial\xi}(p_W,\xi_W) = 0$$

This requires

$$r-\xi_W^{\pm} = \pm \left(\frac{p}{q}\right)_{p_W^{\pm}} (z-z_a)$$

3.11

$$\frac{\partial\tau}{\partial\xi}(p_W,\xi_W) = p_W^{\pm}$$

From 3.8 it is apparent that these conditions are satisfied if

$$W^{\pm} = (p_r, r_a) \qquad 3.12$$

i.e., the geometrical ray parameter and the initial epicentral distance at M_a define the stationary point for both phase functions. Comparing 3.11 with the parametric equation of the straight line ray path in the caustic region

$$\frac{r-r_a}{\frac{\partial\tau}{\partial r_a}} = \frac{z-z_a}{\frac{\partial\tau}{\partial z_a}}$$

we see that

$$\frac{\partial\tau}{\partial z_a} = \pm q_r \quad ; \quad \text{for } \Phi^{\pm} \qquad 3.13$$

This simply means that $\Phi^{\pm}$ correspond to upgoing and downgoing waves respectively.

For high frequencies, the double integral in 3.9 may be evaluated using the n-dimensional stationary phase method

(Appendix 2). After simplification using 3.7, 3.8, 3.10, 3.12 and 3.13 the result is

$$u(r,z) = u(M_a)\, E(W)\, e^{-i\frac{\pi}{4} \operatorname{sig} \overline{\overline{B}}(W) - i\omega \int_{s_a}^{s} \frac{ds}{v}} \qquad 3.14$$

for both upgoing and downgoing waves. Here

$$\overline{\overline{B}}(W) = \begin{bmatrix} \dfrac{-|z-z_a|}{v^2 q_r^3} & -1 \\ -1 & \dfrac{1}{q_r \dfrac{\partial r_a}{\partial \theta}} \end{bmatrix} \qquad 3.15$$

$$E(W) = \left| \frac{r_a}{r \det \overline{\overline{B}}(W)} \right|^{\frac{1}{2}}$$

It follows from 3.15 that

$$E(W) = \left| \frac{r_a \dfrac{\partial r_a}{\partial \theta}}{r \dfrac{\partial r}{\partial \theta}} \right|^{\frac{1}{2}}$$

Since $\cos\theta\, d\theta\, d\phi$ remains constant along the ray in the caustic region, we see that $E(W)$ is in fact related to the ratio of the ray tube cross-sectional areas:

$$E(W) = \left| \frac{d\sigma(r_a, z_a)}{d\sigma(r,z)} \right|^{\frac{1}{2}} \qquad 3.16$$

Now we evaluate the signature of $\bar{\bar{B}}$. At M_c, $|\det \bar{\bar{B}}|$ vanishes. This requires $\partial r_a/\partial\theta < 0$ as is expected for a converging ray tube radiating outwards from the source. Keeping this in mind, the eigenvalues Ω_i of the matrix may be expressed as

$$\Omega_i = \tfrac{1}{2} \left\{ -(c+d) \pm [(c-d)^2 + 4]^{\frac{1}{2}} \right\} \quad ; \quad i = \begin{matrix}1\\2\end{matrix}$$

where

$$c = \left| \frac{\partial r_a}{\partial \theta} \right| \quad ; \quad d = \frac{|z-z_a|}{v^2 q_r^3}$$

At the caustic $c = 1/d$ so that

$$\Omega_1(M_c) = 0 \quad ; \quad \Omega_2(M_c) = -(c+\frac{1}{c}) < 0 \qquad 3.17$$

Now

$$\frac{\partial \Omega_i}{\partial r}(M_c) = \frac{-\cos^2\theta}{2v^2 q_r^3}(1 \pm \gamma) \quad ; \quad i = \begin{matrix}1\\2\end{matrix}$$

where

$$\gamma = \frac{c-\frac{1}{c}}{c+\frac{1}{c}}$$

ranges between zero and unity. Therefore

$$\frac{\partial \Omega_i}{\partial r}(M_c) < 0 \qquad ; \qquad i = 1,2$$

Using this and 3.17 it is clear that Ω_2 remains negative in the vicinity of the caustic while Ω_1 is positive for $r \lesssim r_c$ vanishes at $r = r_c$ and is negative for $r \gtrsim r_c$.

Therefore

$$\text{sig}\,\overline{\overline{B}}(W) = \begin{cases} 0 & , \quad r \lesssim r_c \\ -2 & , \quad r \gtrsim r_c \end{cases} \qquad 3.18$$

Using 3.16 and 3.18 in 3.14 we have

$$u(r,z,\omega) = u(M_a) \left[\frac{d\sigma(M_a)}{d\sigma(r,z)}\right]^{\frac{1}{2}} e^{i\frac{\pi}{2}H(r-r_c) - i\omega \int_{s_a}^{s} \frac{ds}{v}} \qquad 3.19$$

where H is the Heavyside function. Therefore we see there is a $\pi/2$ phase shift as a harmonic wave propagates through a caustic. The condition that det $\overline{\overline{B}}$ vanishes at the caustic uniquely defines the sign of all quantities appearing in $\overline{\overline{B}}$ so that 3.18 always holds for outgoing waves. Therefore a $\pi/2$ phase shift occurs for each caustic encountered along the ray so that the total phase shift due to n_c caustics would be $n_c\pi/2$. For waves propagating inwards toward the source, the same results can be obtained if we let $p \to -p$ in all expressions. It should be emphasized that the sign of the phase shift, positive in this case, is determined solely by the sign convention used in

the Fourier transform pair. If the complex conjugate of 2.2 had been used, the conjugate result, a $-\pi/2$ phase shift for each caustic, would have been obtained.

The ART result for the displacement amplitude is valid only if no caustic is encountered along the ray. We now see that it is verified by 3.19 $(r < r_c)$. However, 3.19 is also valid for regions beyond a caustic. Thus the validity of the ART result can be extended to these regions if a $\pi/2$ phase shift is introduced for each caustic encountered along the ray, i.e., if we replace 1.5 by

$$\vec{u}(M) = S(\omega)U(M)\, e^{in_c\frac{\pi}{2} - i\omega\tau}\, \vec{n}(M) \qquad 3.20$$

where n_c, the number of caustics, is given by 3.5.

CHAPTER 4

RAY AMPLITUDE IN THE VICINITY OF A CAUSTIC

A. Introduction

At a caustic, ART predicts an infinite amplitude for the particle displacement. An infinite amplitude, of course, is physically meaningless and indicates a deficiency in the mathematical formulation of the theory. To obtain the correct dynamic properties of the wave field in the vicinity of a caustic, more exact methods must be used.

In Chapter 2 it was shown that the WKBJ solution in the frequency domain may be expressed formally as an integral over the ray parameter p (equation 2.20). This integral may be evaluated via different approximation techniques. In this chapter, we will first employ the second-order saddle point method (Morse and Feshbach 1953, p. 440) to show that the WKBJ solution is equivalent to the extended ART result 3.20 when the latter is valid. At and near caustics where the second-order saddle point method is inapplicable, the third-order saddle point method (Chester, Friedman and Ursell 1957) will be used to provide the amplitude solution. Then, using a modified version of this method, an improved solution will be obtained. Finally, using the recently developed equal phase method (Chapman 1976, 1978), a time-domain solution valid over the entire spatial domain will be derived.

B. Geometric Ray Approximation

For high frequencies, the second-order saddle point method may be applied to 2.20 to give

$$u(M,\omega) = \frac{(-1)^{\varepsilon} S(\omega) f(p_r)}{\left| r\frac{\partial^2 \tau}{\partial p^2}\right|^{\frac{1}{2}}_{p_r}} \, e^{i\{n\frac{\pi}{2} - \frac{\pi}{4}[\mathrm{sgn}(\frac{\partial^2 \tau}{\partial p^2})_{p_r} + 1]\} - i\omega\tau(p_r,r)} \qquad 4.1$$

where p_r is defined by the saddle point condition

$$\frac{\partial\tau}{\partial p}(p_r) = 0.$$

From the definition of the phase function (equation 2.22) we see that p_r must in fact be the geometrical ray parameter of the ray reaching the observer point. Therefore $\tau_r \equiv \tau(p_r,r)$ is the geometrical arrival time; and the functions G (equations 2.15 and 2.16) reduce to

$$G(M) = (v\rho\cos\theta)^{\frac{1}{2}}_{M}$$

for both P and SV waves.

Using these simplifications and 3.5 for the number of caustics encountered along the ray we can write 4.1 as

$$u(M,\omega) = \frac{\chi(M,\omega,p_r)}{L(M)} \, e^{-i\omega\tau_r + in_c\frac{\pi}{2}}$$

where

$$\chi(M,\omega,p) = (-1)^{\varepsilon} S(\omega) \left| \frac{(v\rho)_{M_O}}{(v\rho)_M} \right|^{\frac{1}{2}} \prod_{j=1}^{k} \left| \frac{(v\rho)_{O_j^-}}{(v\rho)_{O_j^+}} \right|^{\frac{1}{2}} R(O_j)$$

and L(M) is as given in 1.9. This is called the geometrical ray approximation and is identical with the extended ART result 3.20.

C. Airy Approximation

In evaluating the integral 2.20 using the second-order saddle point method, we expanded the phase function $\tau\,(p,r)$ in a Taylor series about the saddle point p_r and limited ourselves to the quadratic term proportional to $\frac{\partial^2\tau}{\partial p^2}\,(p,r)$. This approximation is valid only when $\frac{\partial^2\tau}{\partial p^2}\,(p,r)$ is not too small. This is also clear from the fact that the geometrical ray approximation 4.1 tends to infinity as $\frac{\partial^2\tau}{\partial p^2}(p,r) \to 0$ near a caustic, which is physically meaningless.

For regions near a caustic, we can expand the phase function in a power series in $\xi = p-p_c$ where p_c is the geometrical ray parameter of the ray forming a caustic at the epicentral distance r_c. Since $\frac{\partial^2\tau}{\partial p^2}\,(p_c,r_c) = 0$, the quadratic term in the expansion will be absent. As a result

$$\tau(p,r) = \tau_c + \tau_c'\xi + \frac{1}{6}\tau_c'''\xi^3 + \dots.$$

where

$$\tau_c = \tau(p_c, r)$$

$$\tau_c' = \frac{\partial \tau}{\partial p}(p_c, r) = r - r_c$$

$$\tau_c''' = \frac{\partial^3 \tau}{\partial p^3}(p_c, r) = \frac{-\partial^2 r_c}{\partial p^2}$$

Using up to the cubic term of this expansion, the displacement amplitude 2.20 may be approximated by

$$u(M,\omega) = (-1)^{\varepsilon} S(\omega) \left(\frac{\omega}{2\pi r}\right)^{\frac{1}{2}} e^{-i\omega\tau_c + i(n\frac{\pi}{2} - \frac{\pi}{4})} I \qquad 4.2$$

where

$$I = \int_{-\infty}^{\infty} f(p)\, e^{-i\omega(\tau_c'\xi + \frac{1}{6}\tau_c'''\xi^3)} d\xi$$

Bringing the function f(p) outside the integral sign at the value $p = p_c$ as a slowly varying function and introducing a new variable of integration

$$s = \operatorname{sgn}(\tau_c''') \left| \frac{\omega\tau_c'''}{2} \right|^{\frac{1}{3}} \xi \qquad 4.3$$

we obtain

$$I = \left| \frac{2}{\omega\tau_c'''} \right|^{\frac{1}{3}} f(p_c) \int_{-\infty}^{\infty} Y(y,s)\, ds$$

where

$$Y(y,s) = e^{-i(s^3/3+ys)}$$

$$y = \text{sgn}\,(\tau_c'\tau_c''')\ |\omega\tau_c'|\ \left|\frac{2}{\omega\tau_c'''}\right|^{\frac{1}{3}} \qquad 4.4$$

This last integral is the Airy function Ai (Abramowitz and Stegun 1965) and

$$I = 2\pi \left|\frac{2}{\omega\tau_c'''}\right|^{\frac{1}{3}} f(p_c)\ \text{Ai}(y)$$

Using this together with 2.21 in 4.2, we obtain the displacement in the vicinity of a caustic:

$$u(M,\omega) = \frac{\chi(M,\omega,p_c)}{L(M)}\ e^{-i\omega\tau_c + i(n\frac{\pi}{2} - \frac{\pi}{4})}$$

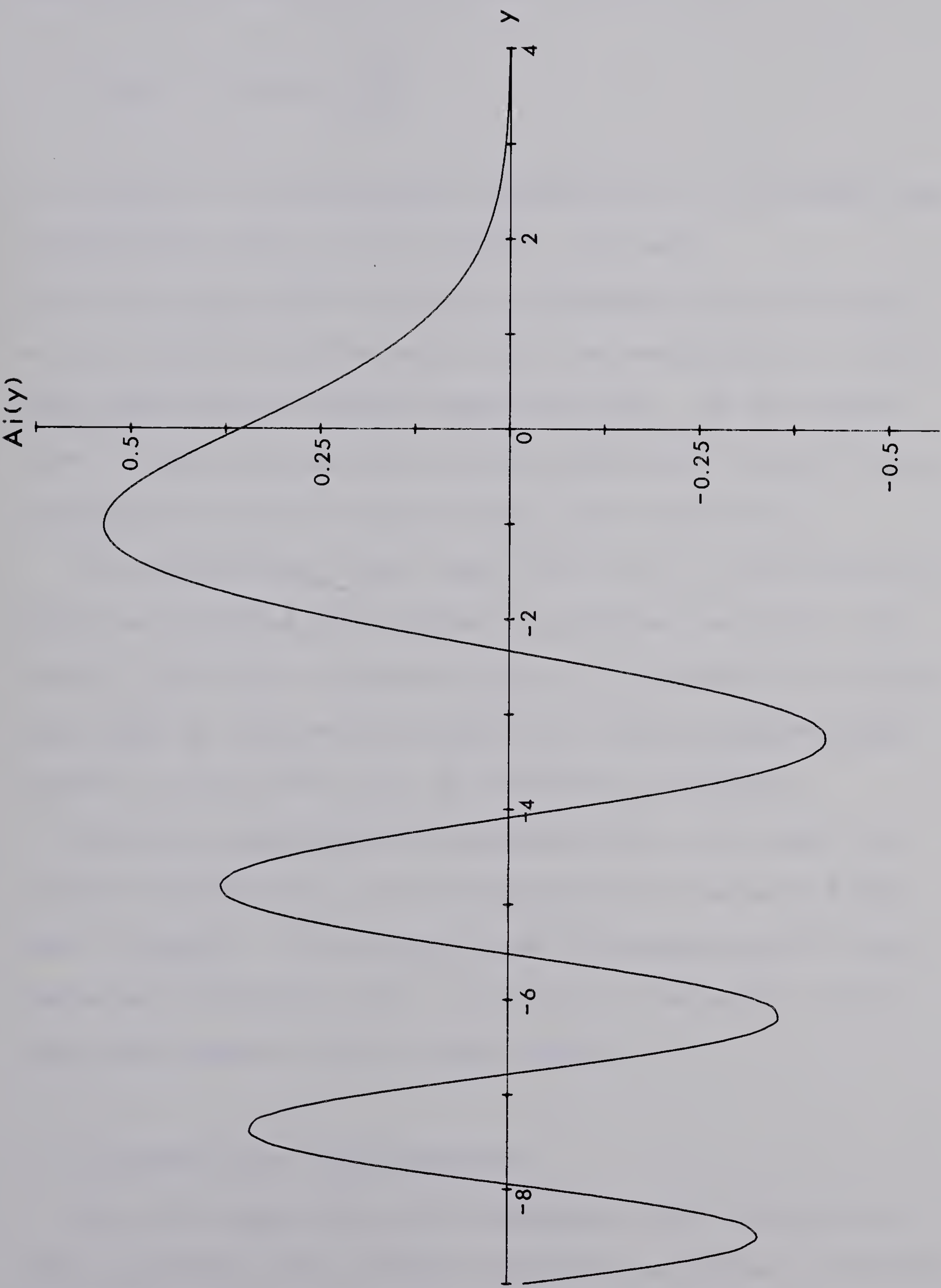
where

$$L(M) = \frac{\left|\dfrac{\partial^2 r_c}{\partial p^2}\right|^{\frac{1}{3}}}{2^{\frac{5}{6}}\omega^{\frac{1}{6}}\ \text{Ai}(y)} \left\{ \left[\frac{r\cos\theta(M)\ \cos\theta_o}{\pi\ v(M_o)\ \sin\theta_o}\right]^{\frac{1}{2}} \prod_{j=1}^{k} \left[\frac{\cos\theta(O_j^+)}{\cos\theta(O_j^-)}\right]^{\frac{1}{2}} \right\}_{p_c}$$

Notice that except when $r = r_c, \tau_c$ is generally not the geometrical arrival time of a ray; furthermore, the phase shift due to caustics is not $n_c\pi/2$ but rather is the average of this value over the two geometrical ray branches forming the caustic.

A graph of the Airy function Ai(y) is shown in Fig. 5 . It decreases monotonically with increasing $|y|$ if $y > 0$,

Figure 5 Graph of the Airy function Ai.


Ai(y)
y
0.5
0.25
0
−0.25
−0.5
4
2
-2
-4
-6
-8

and decreases in an oscillatory manner if $y < 0$. The physical interpretation of this is clear. As

$$\tau_c' \tau_c''' = (r_c - r) \frac{\partial^2 r_c}{\partial p^2}$$

$y > 0$ and $y < 0$ correspond respectively to the shadow and illuminated zones of the caustic. At each point near the caustic on the illuminated side, two rays arrive almost simultaneously and the oscillation is due to the interference between these two rays. On the shadow side no rays are present and the diffracted energy decays exponentially with distance away from the caustic.

The maximum amplitude does not occur at the caustic as might be expected from ART but is shifted away from the caustic into the illuminated zone. In general, the amplitude peak is narrower and has a more pronounced maximum closer to the caustic as the frequency increases.

The Airy approximation presented here has been the standard method for the investigation of the wave field near a caustic. It has been used by Brekhovskikh (1960), Sachs and Silbiger (1970), Cerveny and Zahradnik (1972), Hron and Chapman (1973), among others.

D. Modified Airy Approximation

The airy approximation presupposes that the function $f(p)$ (equation 2.21), which involves the product of reflection, transmission and surface conversion coefficients, is slowly

varying in the neighborhood of the caustic. We now pursue a different approach which avoids this limitation. We will restrict our consideration to rays in the illuminated zone of the caustic ($y < 0$).

Making the change of variable 4.3 but keeping $f(s)$ inside the integral we obtain

$$I = \left| \frac{2}{\omega \tau_c'''} \right|^{\frac{1}{3}} \int_{-\infty}^{\infty} f(s)\, Y(y,s)\, ds \qquad 4.5$$

The saddle points of this integral are

$$s_m = \mp \sqrt{-y} \qquad ; \qquad m = \frac{1}{2}$$

which correspond to

$$p_m = p_c \mp \operatorname{sgn}(\tau_c''') \left| \frac{2\tau_c'}{\tau_c'''} \right|^{\frac{1}{2}} \qquad ; \quad m = \frac{1}{2} \qquad 4.6$$

Thus it follows that

$$r = r_c + \frac{1}{2} \frac{\partial^2 r_c}{\partial p^2} (p_m - p_c)^2 \qquad ; \qquad m = 1,2$$

Comparing this with the Taylor expansion of the geometrical range about $p = p_c$, we see that p_1 and p_2 are in fact the ray parameters of the two geometrical rays arriving at the epicentral distance r. Specifically, we will see later that p_1 and p_2 correspond respectively to the arrival on the direct

and reverse geometrical ray branches which meet at the caustic.

We wish to deform the path of integration in 4.5 from the real s-axis to an alternate contour in the complex s-plane. The function Y(y,s) is everywhere analytic in the s-plane and hence causes no problem. The function f(s), however, involves the reflection and transmission coefficients and terms in $q = (1/v_j^2 - p^2)^{\frac{1}{2}}$ and therefore requires more care. In addition to the branch points at $p_j = \pm \frac{1}{v_j}$, the reflection and transmission coefficients also have poles where the Rayleigh denominator vanishes. These are the Rayleigh poles which are always real and positive (Archenbach 1973). In general, we see that all the poles of f(p) are real and satisfy the inequality $| p_j | \geq 1/v_{max}$ where v_{max} is the maximum velocity encountered at interfaces. The branch cuts for the square roots can be taken to be on the real p-axis (Fig. 6). The saddle points, branch cuts and poles in the complex s-plane are shown in Fig. 7.

Defining the contours of integration to be those as shown in Fig. 7 and realizing that the most significant contributions come from the saddle points, we obtain from 4.5.

$$I = \left| \frac{2}{\omega \tau_c'''} \right|^{\frac{1}{3}} \left\{ f(p_1) \int_{\Gamma_1} Y(y,s)ds + f(p_2) \int_{\Gamma_2} Y(y,s)ds \right\}$$

If the endpoints of the new contours are kept within the shaded zones in Fig. 7, the integrals can be identified

Figure 6   Poles and branch cuts on the p-plane for the integral 4.5.

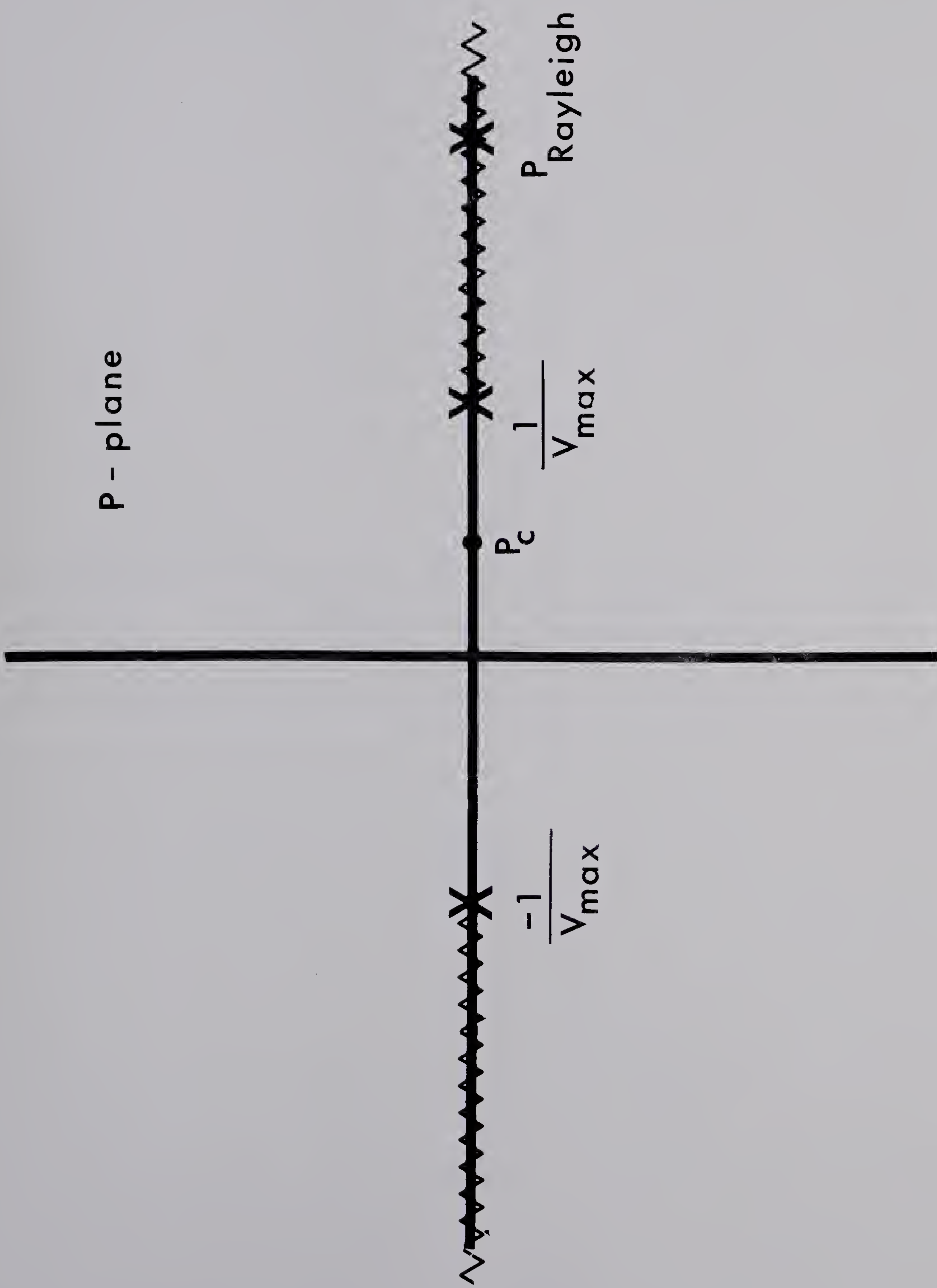
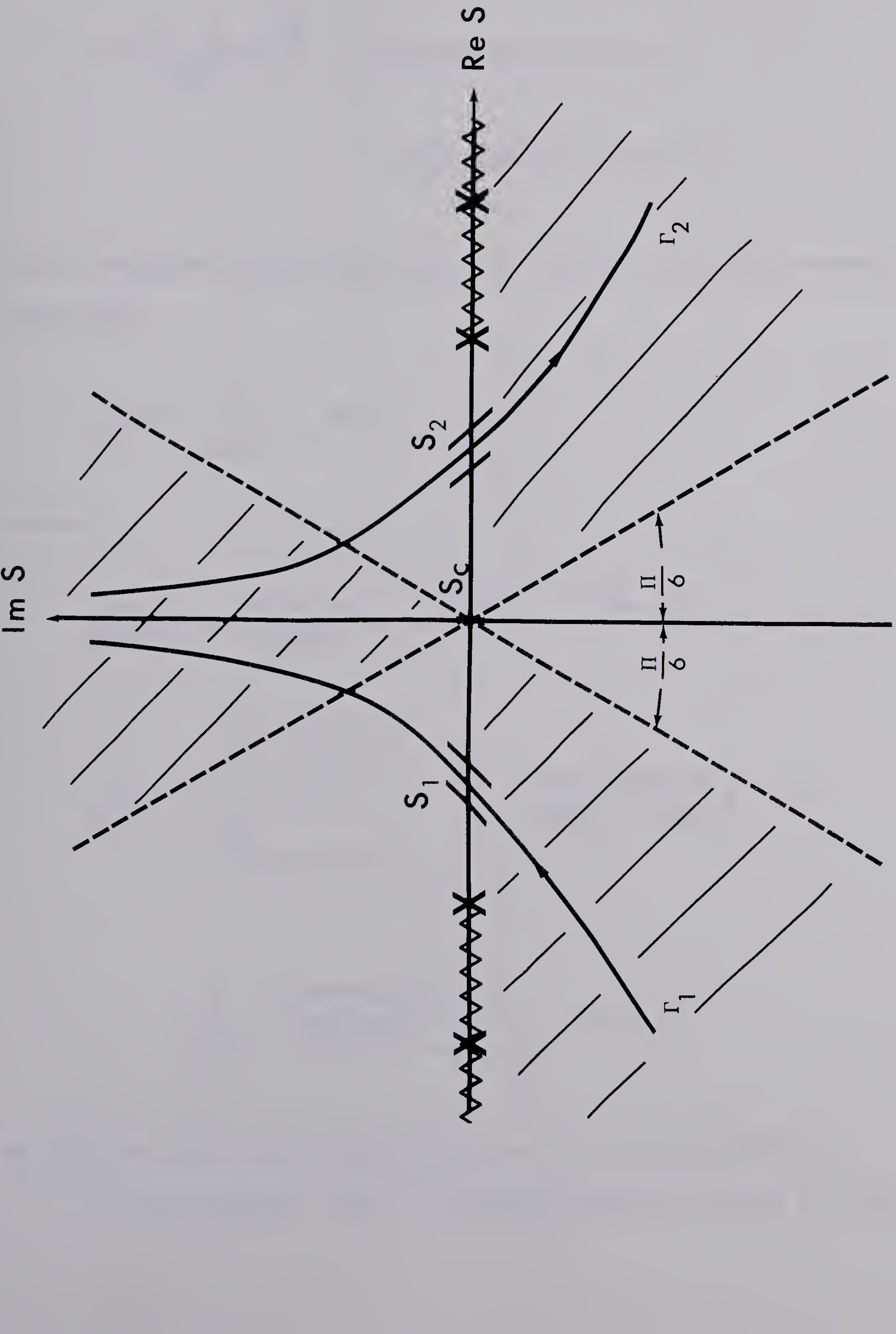

P - plane
P_Rayleigh
1/V_max
P_c
-1/V_max

Figure 7 Poles, branch cuts, saddle points and contours on the s-plane for the integral 4.5.


Re S
Im S
S_c
S_1
S_2
Γ_1
Γ_2
$\frac{\Pi}{6}$
$\frac{\Pi}{6}$

as linear combinations of the Airy functions Ai and Bi (Appendix 3) and

$$I = \left| \frac{2}{\omega\tau_c'''} \right|^{\frac{1}{3}} \pi \left\{ f(p_1)[Ai(y)+iBi(y)] \right.$$

$$\left. + f(p_2)\ [Ai(y)-iBi(y)] \right\}$$

This, together with 2.21 and 4.2, yields the displacement amplitude

$$u(M,\omega) = \sum_{m=1}^{2} u_m\ (M,\omega) \qquad 4.7$$

where

$$u_m(M,\omega) = \frac{\chi(M,\omega,p_m)}{L_m(M)}\ e^{-i\omega\tau_c + i(n\frac{\pi}{2} - \frac{\pi}{4})}$$

$$L_m(M) = \frac{2^{\frac{1}{6}} \left| \dfrac{\partial^2 r_c}{\partial p^2} \right|^{\frac{1}{3}}}{\omega^{\frac{1}{6}}[Ai(y) \pm Bi(y)]} \left\{ \left[\frac{r\cos\theta(M)\cos\theta_o}{\pi v(M_o)\ \sin\theta_o} \right]^{\frac{1}{2}} \right.$$

$$\left. \prod_{j=1}^{k} \left[\frac{\cos\theta(O_j^{+})}{\cos\theta(O_j^{-})} \right]^{\frac{1}{2}} \right\}_{p_m} \quad ;\ m = \begin{matrix} 1 \\ 2 \end{matrix}$$

We denote this result the modified Airy approximation.

If the function f(p) is indeed slowly varying in the

vicinity of the caustic, we can set $f(p_1) = f(p_2) = f(p_c)$ and 4.7 reduces to the Airy approximation as is expected. It is clear that the modified Airy approximation consists of contributions from both ray branches. The linear combination $Ai(y) \mp iBi(y)$ can be expressed respectively as Hankel functions of one-third order of the first and second kinds (Abramowitz and Stegun (1965)) which, for $|y| >> 1$, can be expanded asymptotically in 4.7 to yield

$$u_m(M,\omega) = \frac{\chi(M,\omega,p_m)}{L_m(M)} e^{-i\omega\tau_m + in_m \frac{\pi}{2}} \qquad 4.8$$

where

$$L_m(M) = |2\tau_c'\tau_c'''|^{\frac{1}{4}} \left\{ \left[\frac{r\cos\theta(M)\cos\theta_o}{v(M_o)\sin\theta_o}\right]^{\frac{1}{2}} \prod_{j=1}^{k} \left[\frac{\cos\theta(O_j^+)}{\cos\theta(O_j^-)}\right]^{\frac{1}{2}} \right\}_{p_m}$$

$$n_m = n - \frac{1}{2}(1 \mp 1), \qquad m = \begin{matrix}1\\2\end{matrix} \qquad 4.9$$

and

$$\tau_m = \tau_c \pm \frac{\omega|\tau_c'''|}{3} \left|\frac{2\tau_c'}{\tau_c'''}\right|^{\frac{3}{2}} \quad ; \quad m = \begin{matrix}1\\2\end{matrix}$$

$$= \tau_c + \tau_c' (p_m - p_c) + \frac{\tau_c'''}{6} (p_m - p_c)^3 \qquad 4.10$$

are the arrival times of the two geometrical rays.

From 4.6 and 4.10 we see that p_1 always corresponds to the ray on the direct travel time branch ($\partial r/\partial p > 0$) and p_2

the ray on the reverse travel time branch ($\frac{\partial r}{\partial p} < 0$).
The ray on the reverse branch always arrives earlier than the one on the direct branch. Thus, we can conclude that n_m in 4.9 is in fact the number of caustics n_c (equation 3.5) encountered by the ray. Furthermore, it can be shown that for $p_m \simeq p_c$:

$$|2\tau_c'\tau_c'''|^{\frac{1}{2}} \simeq \left|\frac{\partial r}{\partial p}\right|_{p_m}$$

so that 4.8 in fact are the geometrical ray amplitudes. Thus, at sufficiently large distances from the caustic in the illuminated zone, the modified Airy approximation is equivalent to the superposition of the two geometrical ray amplitudes.

Using the same procedures, the Airy approximation can be shown to consist of two terms similar to 4.8, except that both terms are evaluated at $p = p_c$ and therefore have the same absolute magnitudes. This explains the complete destructive interference that occurs at epicentral distances where the Airy function vanishes. More importantly, this also shows that the Airy approximation is likely to be accurate only if the two geometrical ray branches have comparable amplitudes near the caustic. For wave propagation in layered media, this condition is often not fulfilled. In the vicinity of the PKKP caustic, for instance, the amplitude of one ray branch drops sharply to zero due to a

zero in the reflection coefficient while the other branch remains at a relatively large amplitude (Richards 1973). In such cases the modified Airy approximation is expected to be more accurate than the Airy approximation as contributions from the two ray branches are given proper weighting. Numerical results illustrating this point will be presented in the next chapter.

The modified Airy approximation, however, is not without limitations. Since we have assumed in its derivation that $\frac{\partial r}{\partial p} \simeq 0$, or equivalently

$$|r-r_c| \simeq 0 \qquad 4.11$$

it is valid only in the neighborhood of a caustic. We now determine the size of the region in which it is applicable.

In general, the asymptotic expressions for Hankel functions are valid if the value of the argument is comparable with or greater than the order of the Hankel function. In our case, this means that 4.8 follows from 4.7 provided $\beta \gtrsim 1$ where

$$\beta = \frac{\omega |r-r_c|^{\frac{3}{2}}}{\left|\frac{1}{2}\,\frac{\partial^2 r}{\partial p^2}\right|^{\frac{1}{2}}_{p_c}}$$

For high frequencies, there exists a region of epicentral distances defined by $\Delta r_{min} < |r - r_c| < \Delta r_{max}$ within

which both this condition and 4.11 are satisfied simultaneously. Δr_{min} is the value of $|r - r_c|$ at which $\beta = 1$ while Δr_{max} is the limit beyond which 4.11 becomes a poor approximation. Within this region, the geometrical ray and modified Airy approximations are equivalent and both results are valid.

For $|r - r_c| > \Delta r_{max}$, the geometrical ray result is certainly applicable as the caustic is now even farther away. For $|r - r_c| < \Delta r_{min}$, the modified Airy approximation can be used as r in getting closer to the caustic However, $\beta < 1$ in this region and the geometrical ray approximation is invalid as it is no longer equivalent to the modified Airy approximation. In general, for high frequencies, we conclude that the modified Airy and geometrical approximations should be used when $\beta < 1$ and $\beta > 1$ respectively.

E. WKBJ Seismogram

We now consider an alternative method developed by Chapman (1976,1978) for the evaluation of the integral in 2.20. Although the result will not be as simple for numerical computation as those previously presented, it has the advantage of being valid for all regions so that only one expression need be used at all times. In the next chapter, it will be used as a check for the accuracy of the previous results.

We invert the Fourier transform in 2.20 to obtain

$$u(M,t) = F(r,t) * \int_{-\infty}^{\infty} f(p)\ \delta[t-\tau(p,r)]$$

where

$$F(r,t) = \frac{(-1)^{\varepsilon}}{\pi(2r)^{\frac{1}{2}}}\ e^{i(n-1)\frac{\pi}{2}}\ \frac{dS(t)}{dt} * \frac{H(t)}{t^{\frac{1}{2}}}$$

and * denotes a convolution. Convolving this with a boxcar function of width Δt:

$$B(t,\Delta t) = \frac{1}{\Delta t}\ [H(t/\Delta t) - H(t/\Delta t-1)]$$

we obtain the smoothed time response

$$\langle u(M,t)\rangle_{\Delta t} = F(r,t) * \sum_i \frac{\Delta p_i}{\Delta t}\ \langle f(p)\rangle_{\Delta p_i}$$

where

$$\Delta p_i = |p_i^+ - p_i|$$

p_i and p_i^+ being the solutions of the equal phase conditions

$$t = \tau(p,r)$$

and

$$t + \Delta t = \tau(p,r)$$

respectively; and

$$\langle f(p) \rangle_{\Delta p_i} = \frac{1}{\Delta p_i} \int_{p_i^-}^{p_i^+} f(p)\,dp$$

is the average value of f(p) over the interval Δp_i. This is called the WKBJ seismogram.

Depending on whether the behaviour of the phase function is essentially quadratic or cubic in p in the neighborhood of interest, we will see in the next chapter that the WKBJ seismogram is equivalent to the geometrical ray and modified Airy approximations. However, its numerical evaluation is more complicated due to the convolution intergral.

CHAPTER 5

NUMERICAL RESULTS

A. Introduction

This chapter presents numerical results based on the extended ART. First, the various methods used to interpolate the velocity data are discussed. The corresponding results for the geometrical range, travel time, and the derivatives $\frac{\partial r}{\partial p}$ and $\frac{\partial^2 r}{\partial p^2}$ are presented in Appendices 4, 5, and 6. The accuracy of the Airy and modified Airy approximations is next compared. Finally, synthetic seismograms for realistic models are shown.

B. Interpolation of the Velocity Data

We introduce a cylindrical coordinate system (r,z,ϕ) where z, the depth, is positive downwards. As before, the azimuthal angle ϕ may be ignored. Consider a section of the ray specified by the ray parameter p, situated at depths between z_i and z_{i+1}. If we denote the travel time of this section of the ray by t_i and the corresponding range of horizontal distance by r_i, then the ray integrals are given by

$$t_i = \int_{z_i}^{\hat{z}_i} \frac{dz}{va} \quad ; \quad r_i = \int_{z_i}^{\hat{z}_i} \frac{pv}{a}\, dz$$

where $\hat{z}_i = z_{i+1}$ if the ray has no turning point and $\hat{z}_i = z_i^*$ if the ray has a turning point at (r_i^*, z_i^*); and

$$a = (1 - p^2v^2)^{\frac{1}{2}}$$

These integrals define the kinematical properties of the ray. To determine the ray amplitude, we also must compute the derivatives $\frac{\partial r}{\partial p}$ and $\frac{\partial^2 r}{\partial p^2}$ (near a caustic). We denote these four quantities the ray characteristics.

In general, the velocity data consist of a discrete set of velocities at given depths. The data must be interpolated to provide the velocity function required for the evaluation of the ray characteristics. The simplest approach is to assume that the velocity varies linearly between any two consecutive data points. For this case, analytic expressions may be obtained for all ray characteristics (Appendix 4). However, piece-wise linear interpolation introduces false interfaces of second order at each data point where the velocity gradient is discontinuous. This tends to give rise to anomalous behaviour of the ray amplitudes.

To avoid this difficulty, the velocity data may be fitted with a smoothed spline of the form

$$v(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$

(Chapman 1971) which ensures the continuity of the velocity gradient as well as its derivative. This formula can be used insofar as the data can be grouped into sets which can be approximated by piece-wise continuous functions.

With this interpolation, however, the ray characteristics must be evaluated by numerical integration (Appendix 5).

Alternatively, we may apply the smoothed spline approximation to interpolate the function $z=z(v)$ instead of $v=v(z)$ (Cerveny 1977), i.e.

$$z(v) = b_0 + b_1 v + b_2 v^2 + b_3 v^3 \qquad 5.1$$

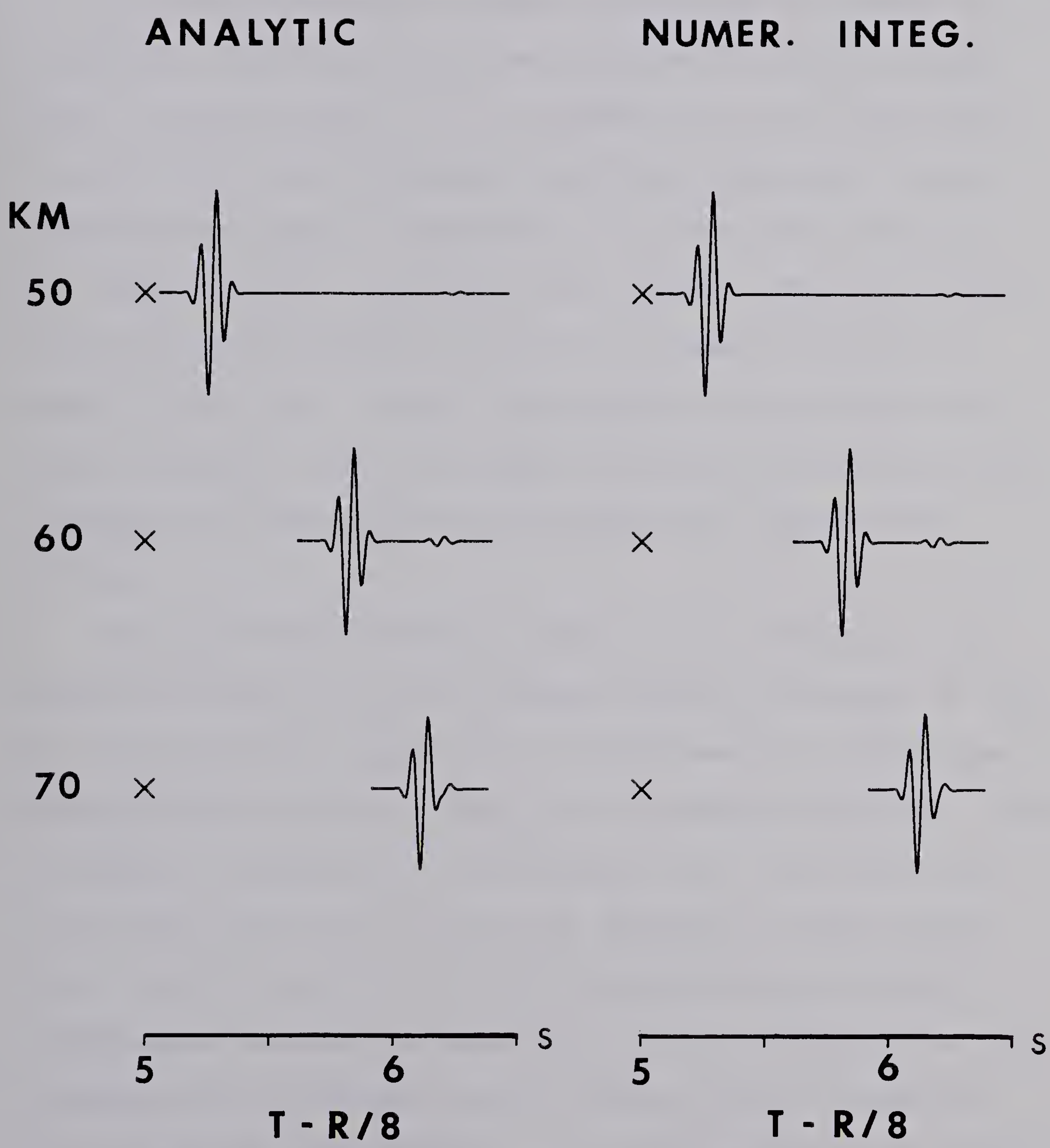
This formula is applicable if the data can be grouped into sets which can be approximated by piece-wise monotonic functions. To compute the ray characteristics for this case, it is more convenient to use v as the intergration variable:

$$t_i = \int_{v_i}^{\hat{v}_i} \frac{1}{va}\frac{dz}{dv}\,dv \quad ; \quad r_i = \int_{v_i}^{\hat{v}_i} \frac{pv}{a}\frac{dz}{dv}\,dv \qquad 5.2$$

where $\hat{v}_i = v_{i+1}$ if the ray has no turning point and $\hat{v}_i = v_i^*$ if the ray has a turning point at (r_i^*, z_i^*). With the smoothed spline interpolation 5.1, it is clear from 5.2 that analytic expressions in closed form may be obtained for the ray characteristics (Appendix 6). This avoids the necessity for numerical integration.

Fig. 8 compares the primary P ray seismograms obtained using the two different smoothed spline interpolations to the velocity function $v(z) = 4e^{0.0277z}$. The agreement between the two are excellent. However, the cost in CPU time for numerical integration with $v=v(z)$ is about twice

Figure 8 Comparison of seismograms computed using numerical integration with $v=v(z)$ and analytic expressions derived from $z=z(v)$. The axis displays the reduced time where R is the epicentral distance and a reducing velocity of 8 km/sec has been used.


ANALYTIC
NUMER. INTEG.
KM
50
60
70
5
6
S
T - R/8
5
6
S
T - R/8

that for the analytic expressions derived with $z = z(v)$.

C. Comparisons of the Airy and Modified Airy Approximations

We first consider a simple model used by Cerveny and Zahradnik (1972) for the investigation of the wave field near a caustic (Fig. 9). It consists of a top layer with a piece-wise linear velocity structure overlying a homogeneous half space. The ratio of P and S wave velocities is constant ($=\sqrt{3}$) throughout the medium. The ray diagram and travel time-distance curves for primary P rays are shown in Fig. 10. Branch 1 corresponds to reflected rays while branches 2 and 3 represent refracted rays which form a caustic on the surface at an epicentral distance of 120 km.

The amplitude-distance curves of the refracted rays near the caustic found by various methods are shown in Fig. 11. The vertical component of displacement is considered. The amplitude curves A_1 and A_2 are computed using ART. They correspond respectively to branches 2 and 3 of the travel time curve and they both tend to infinity at the caustic. Away from the caustic, they are comparable and remain essentially constant in magnitude. This indicates that the function $f(p)$ is indeed slowly varying near the caustic. $\tilde{A}_a$ is the Airy approximation. It increases from very small values in the shadow zone, remains finite at the caustic, and reaches a maximum beyond the caustic in the illuminated zone. $\tilde{A}_m$ is the modified Airy approximation. It is almost

Figure 9 Velocity models for the study of caustics - the Cerveny and modified Cerveny models.

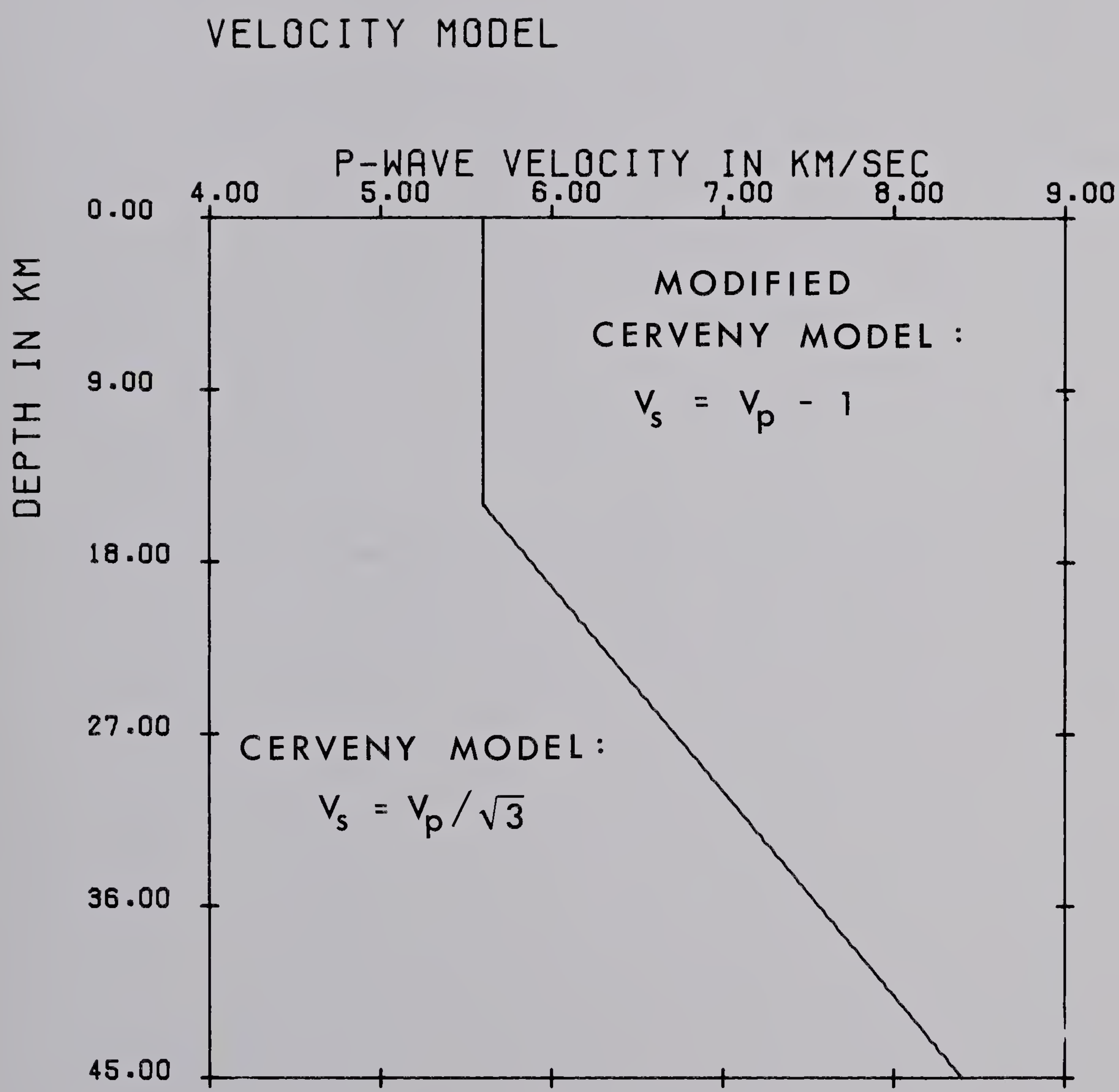

VELOCITY MODEL
P-WAVE VELOCITY IN KM/SEC
4.00
5.00
6.00
7.00
8.00
9.00
DEPTH IN KM
0.00
9.00
18.00
27.00
36.00
45.00
MODIFIED CERVENY MODEL :
$V_s = V_p - 1$
CERVENY MODEL:
$V_s = V_p/\sqrt{3}$

Figure 10 Ray diagrams and travel time-distance curves of primary P rays for the Cerveny model.

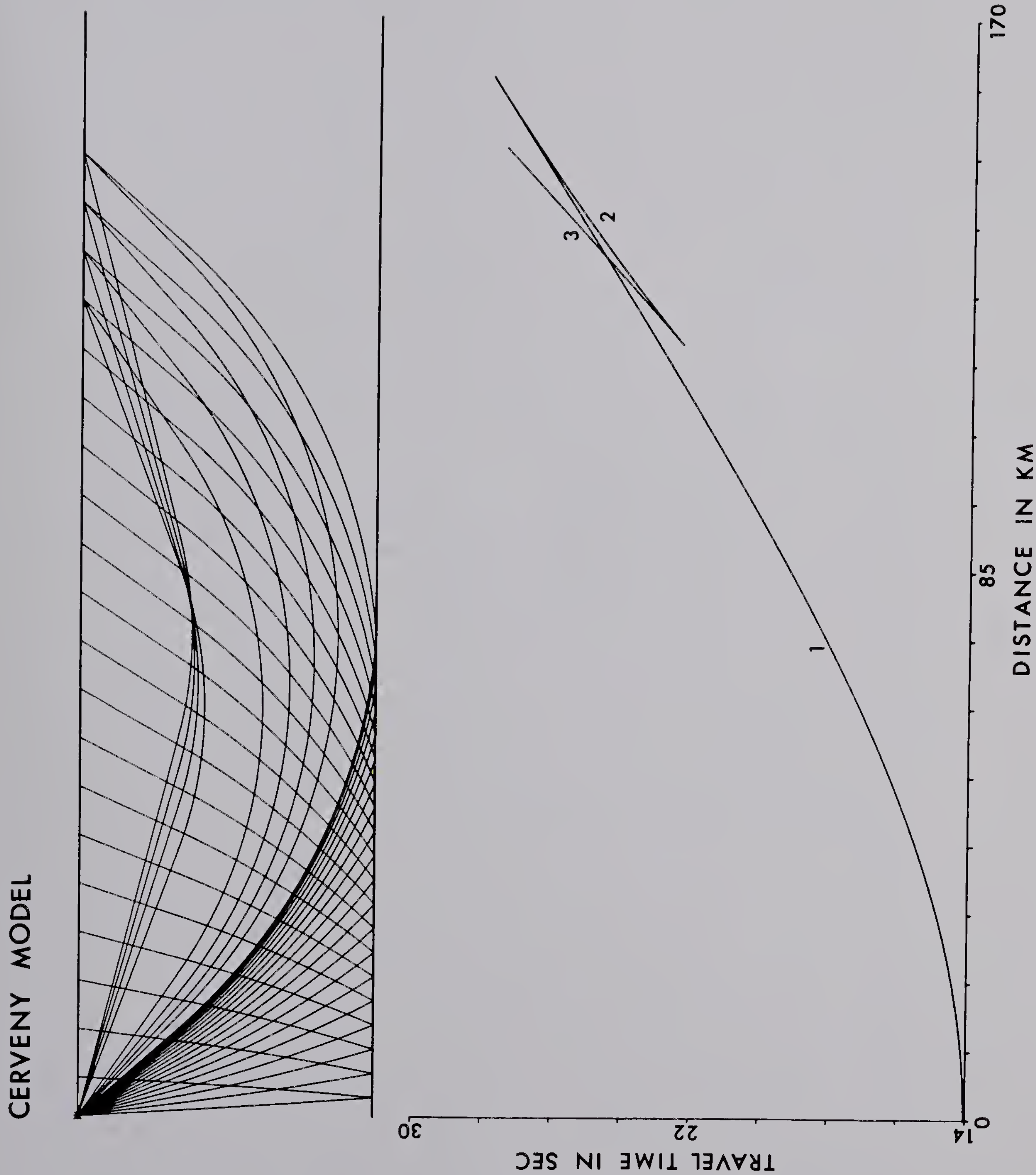
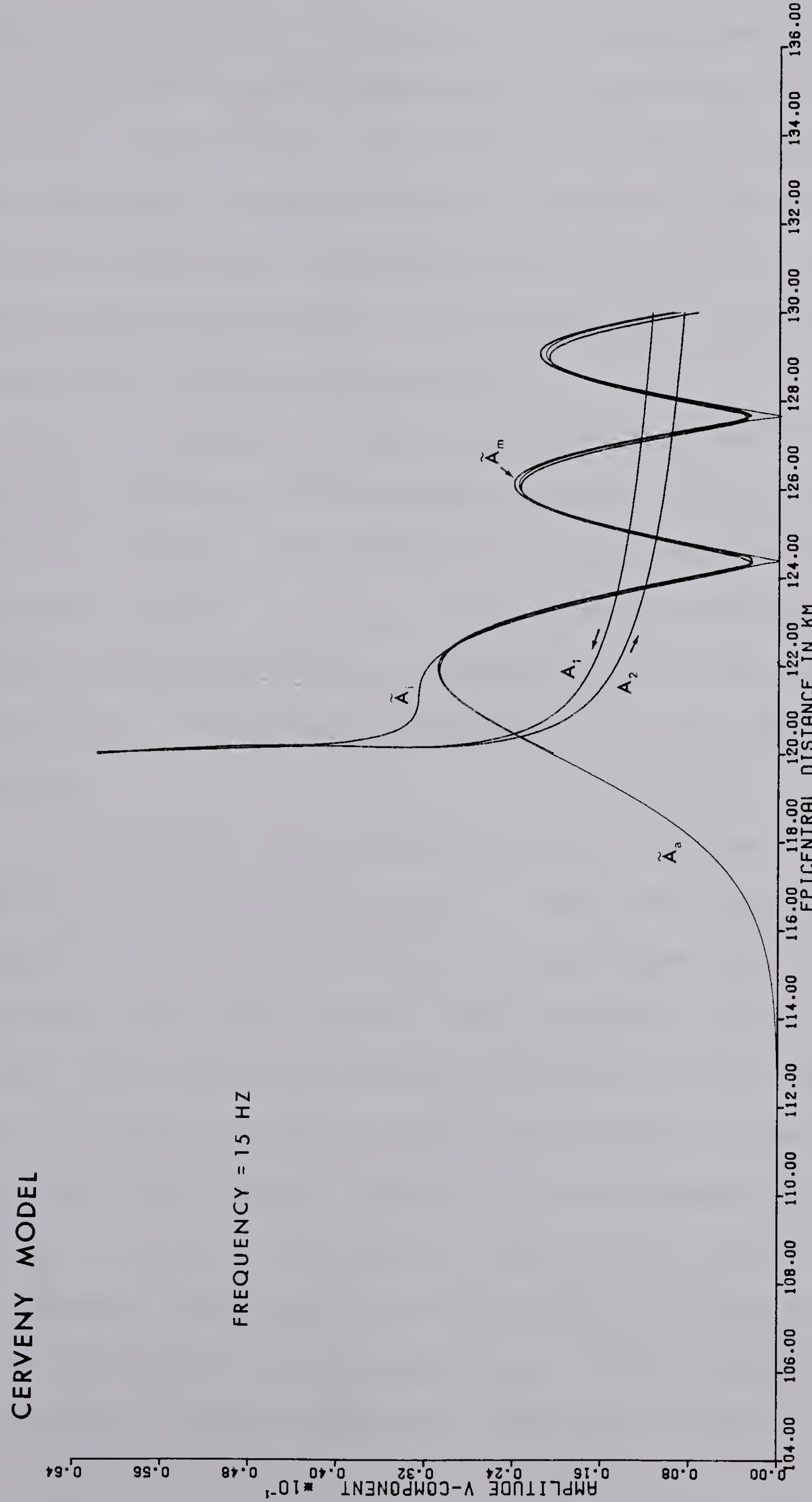

CERVENY MODEL
TRAVEL TIME IN SEC
30
22
14
0
85
170
DISTANCE IN KM
1
2
3

Figure 11 Amplitude-distance curves of refracted P rays for the Cerveny model. A_1 and A_2 are the geometrical ray amplitudes of the two ray branches. The arrows indicate the direction of increasing ray parameter. $\tilde{A}_i$ is the interference amplitude of these two branches. $\tilde{A}_a$ is the Airy approximation and $\tilde{A}_m$ is the modified Airy approximation.

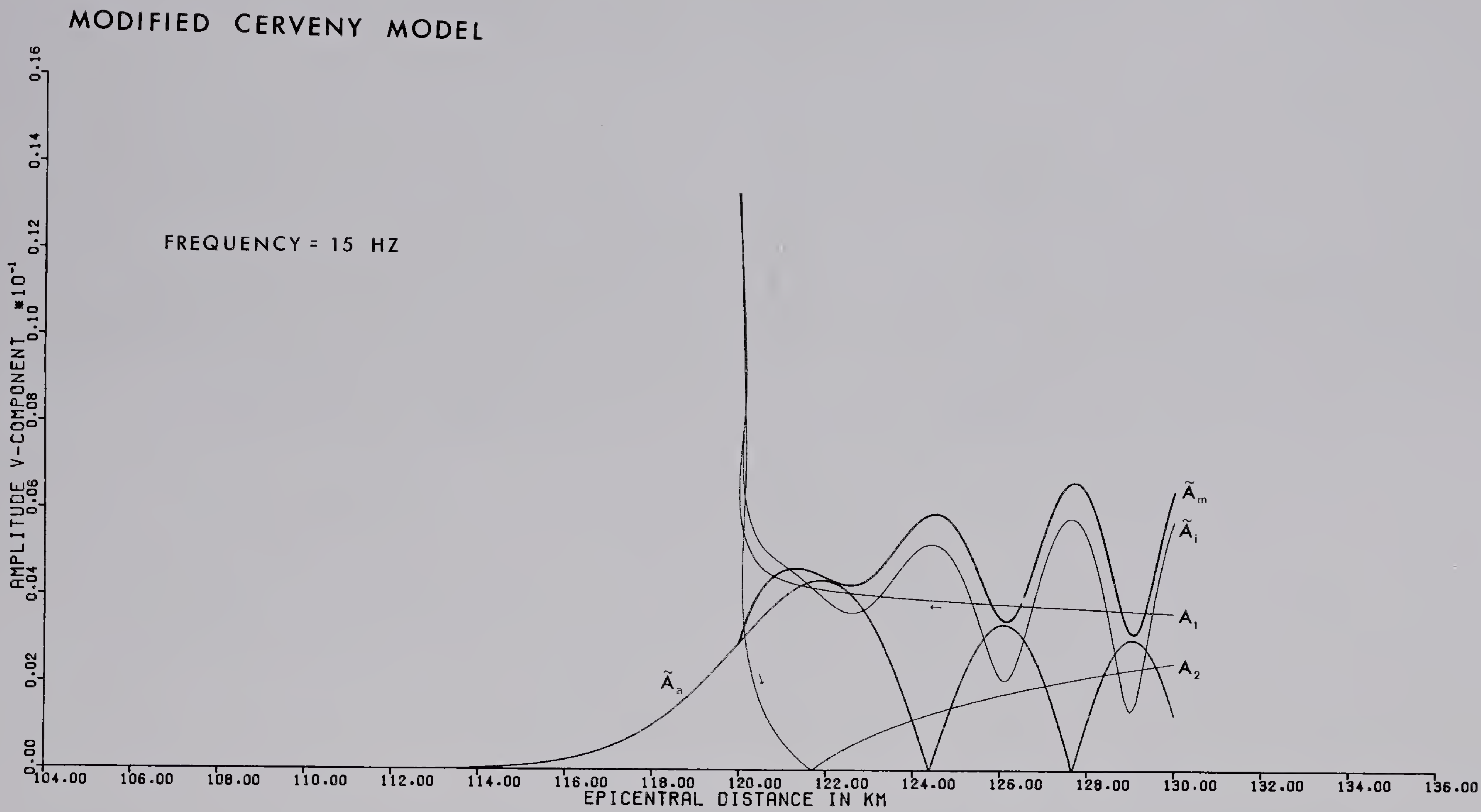

CERVENY MODEL
FREQUENCY = 15 HZ
AMPLITUDE V-COMPONENT *10⁻¹
EPICENTRAL DISTANCE IN KM
0.00
0.08
0.16
0.24
0.32
0.40
0.48
0.56
0.64
104.00
106.00
108.00
110.00
112.00
114.00
116.00
118.00
120.00
122.00
124.00
126.00
128.00
130.00
132.00
134.00
136.00
$\tilde{A}_a$
$\tilde{A}_i$
$\tilde{A}_m$
A_1
A_2

identical with the Airy approximation over the entire illuminated zone as is expected for a slowly varying f(p). $\tilde{A}_i$ is the interference amplitude of the geometrical branches. Note that $\tilde{A}_a$, $\tilde{A}_m$ and $\tilde{A}_i$ all give practically the same amplitudes in some range of epicentral distances beyond the caustic. Immediately near the caustic ART is inapplicable and the Airy and modified Airy approximations provide more accurate amplitudes.

Next we consider a model which has the same P wave velocity structure as the previous model. The S wave velocity, however, now differs from the P wave velocity by a constant amount ($v_s = v_p - 3.6$). The kinematic properties of the primary P rays, therefore, are identical with those above. The dynamic properties, however, are now different.

Fig. 12 shows the amplitude-distance curves of the refracted rays near the caustic. Note that the two geometrical ray branches A_1 and A_2 no longer have comparable amplitudes near the caustic. The branch A_2 drops sharply to zero just beyond the caustic due to a zero in the surface conversion coeffient while the branch A_1 remains at relatively high values. Thus f(p) is no longer slowly varying. Except at distances close to the caustic, the interference refracted wave amplitude $\tilde{A}_i$ matches closely the modified Airy approximation $\tilde{A}_m$. The Airy approximation $\tilde{A}_a$, however, yields completely different results.

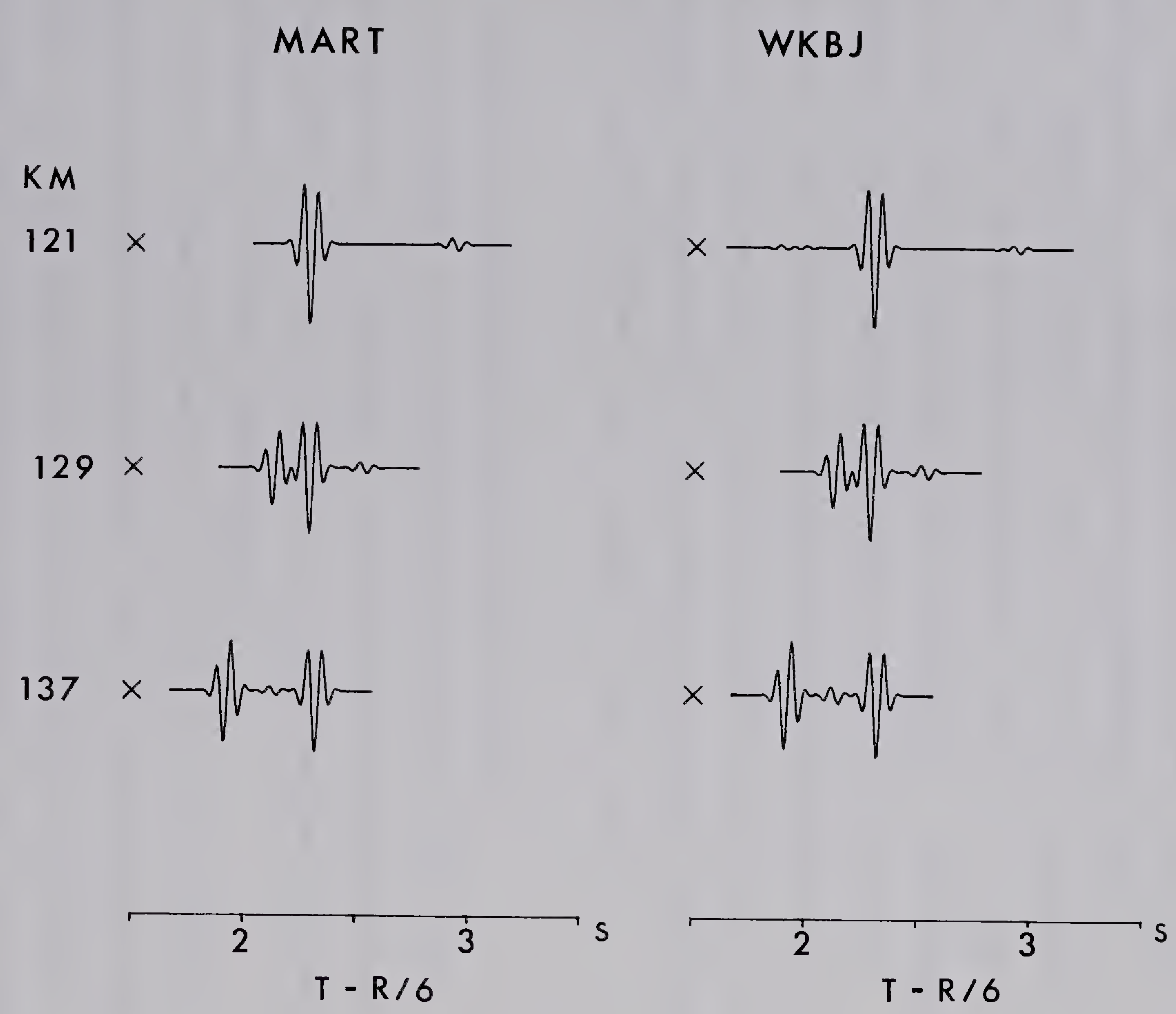
Figure 12 Amplitude-distance curves of refracted P rays for the modified Cerveny model. A_1 and A_2 are the geometrical ray amplitudes of the two ray branches. The arrows indicate the direction of increasing ray parameter. $\tilde{A}_i$ is the interference amplitude of these two branches. $\tilde{A}_a$ is the Airy approximation and $\tilde{A}_m$ is the modified Airy approximation.



MODIFIED CERVENY MODEL
FREQUENCY = 15 HZ
AMPLITUDE V-COMPONENT *10⁻¹
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
104.00
106.00
108.00
110.00
112.00
114.00
116.00
118.00
120.00
122.00
124.00
126.00
128.00
130.00
132.00
134.00
136.00
EPICENTRAL DISTANCE IN KM
$\tilde{A}_m$
$\tilde{A}_i$
A_1
A_2
$\tilde{A}_a$

Figure 13 Comparison of MART and WKBJ seismograms for the modified Cerveny model. The axis displays the reduced time where R is the epicentral distance and a reducing velocity of 6 km/sec has been used.

MODIFIED CERVENY MODEL



To show that the modified Airy approximation is indeed more accurate than the Airy approximation in this case, the results are compared with WKBJ seismograms (Fig. 13). On the left are synthetic seismograms calculated using ART augmented with the modified Airy approximation near a caustic (MART seismograms). The modified Airy approximation is used for the MART seismogram at 121 km. On the right are the corresponding WKBJ seismograms. The excellent agreement between the MART and WKBJ seismograms clearly shows that the modified Airy approximation is superior to the Airy approximation in situations where the product of reflection, transmission, and surface conversion coefficients varies significantly in the vicinity of the caustic.

D. Synthetic Seismograms for Realistic Models

The FORTRAN IV program developed by Hron (1971) for the computation of synthetic seismograms for layered media was extended using the formulae developed earlier to include the effects of vertical inhomogeneity.

Fig. 14 shows synthetic seismograms for the vertical component of surface displacement for the Mississippi model in Fig. 15. The major arrivals are identified and correlated over the various epicentral distances. Fig. 16 shows similar synthetic seismograms for a modified version of the Mississippi model. Here the inhomogeneous layers have been replaced by homogeneous layers such that the vertical

Figure 14 Synthetic seismograms for the Mississippi model. The axis displays the reduced time where R is the epicentral distance and a reducing velocity of 4 km/sec has been used.

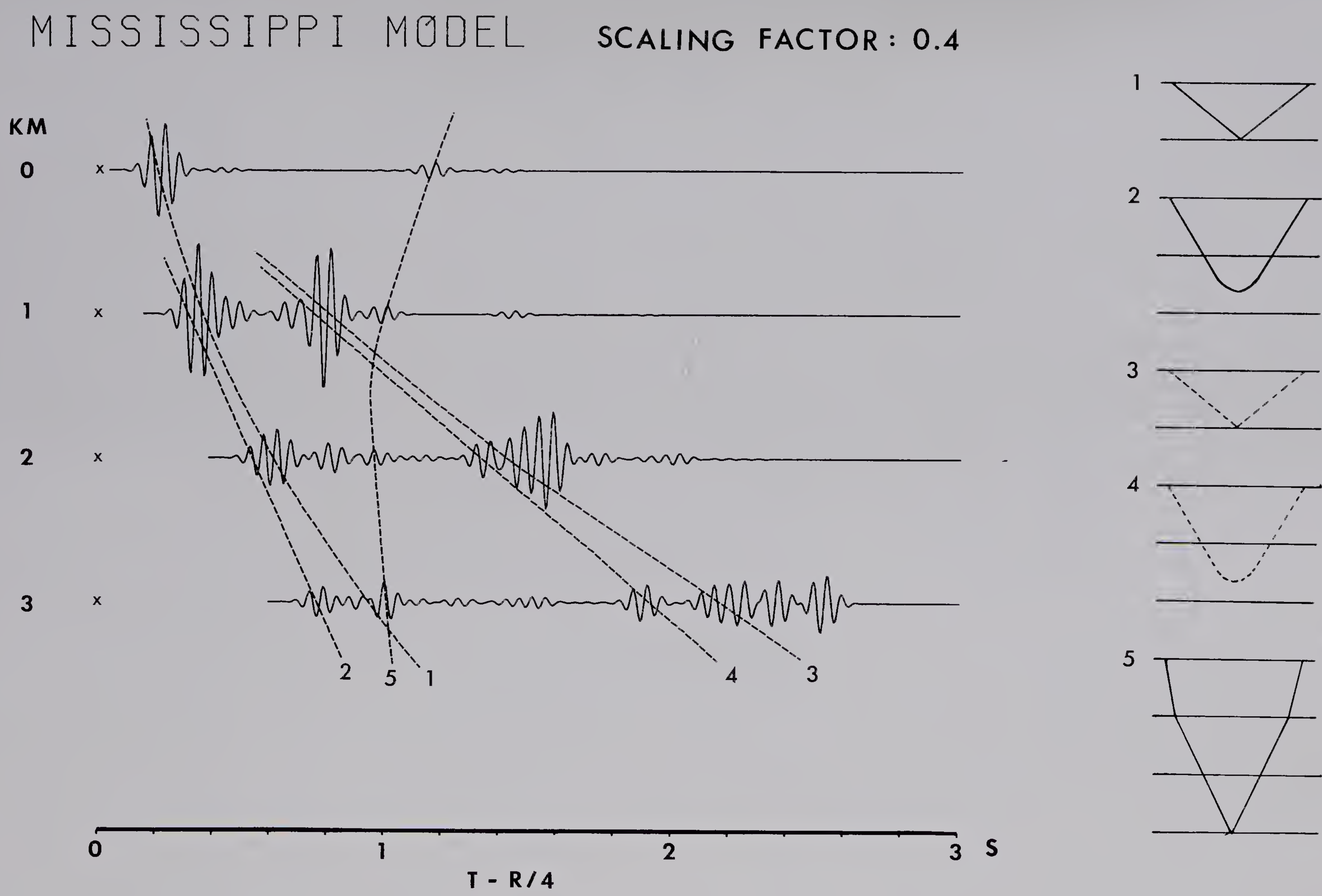

MISSISSIPPI MØDEL
SCALING FACTOR : 0.4
KM
0
1
2
3
2
5
1
4
3
0
1
2
3
S
T - R/4
1
2
3
4
5

Figure 15 The Mississippi model. The model, based on actual well survey (Narvarte 1947), is typical of certain parts of Mississippi. The chained lines show a modified version of the model where the inhomogeneous layers are replaced by homogeneous layers such that the vertical travel times are preserved.

MISSISSIPPI MODEL

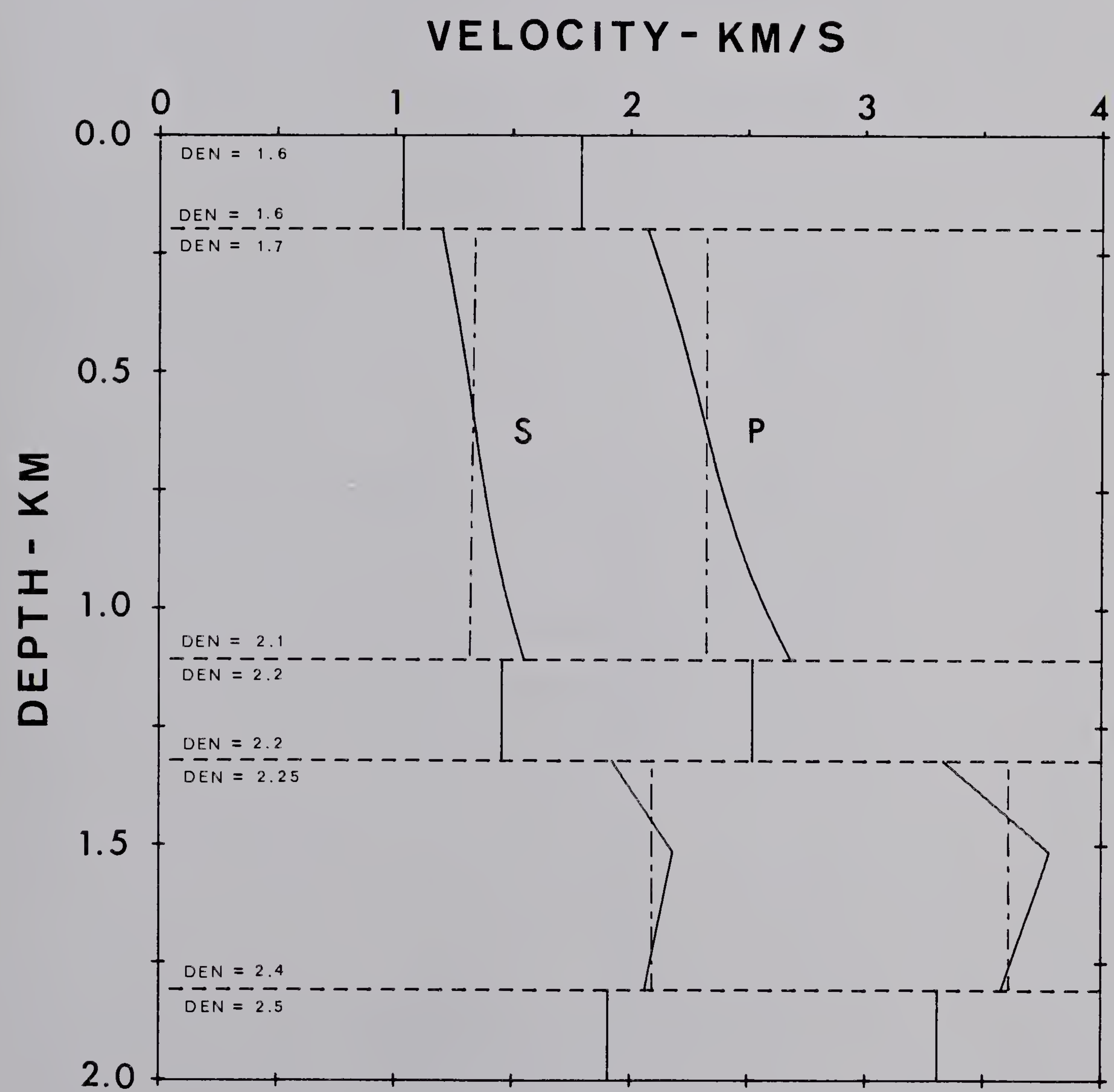
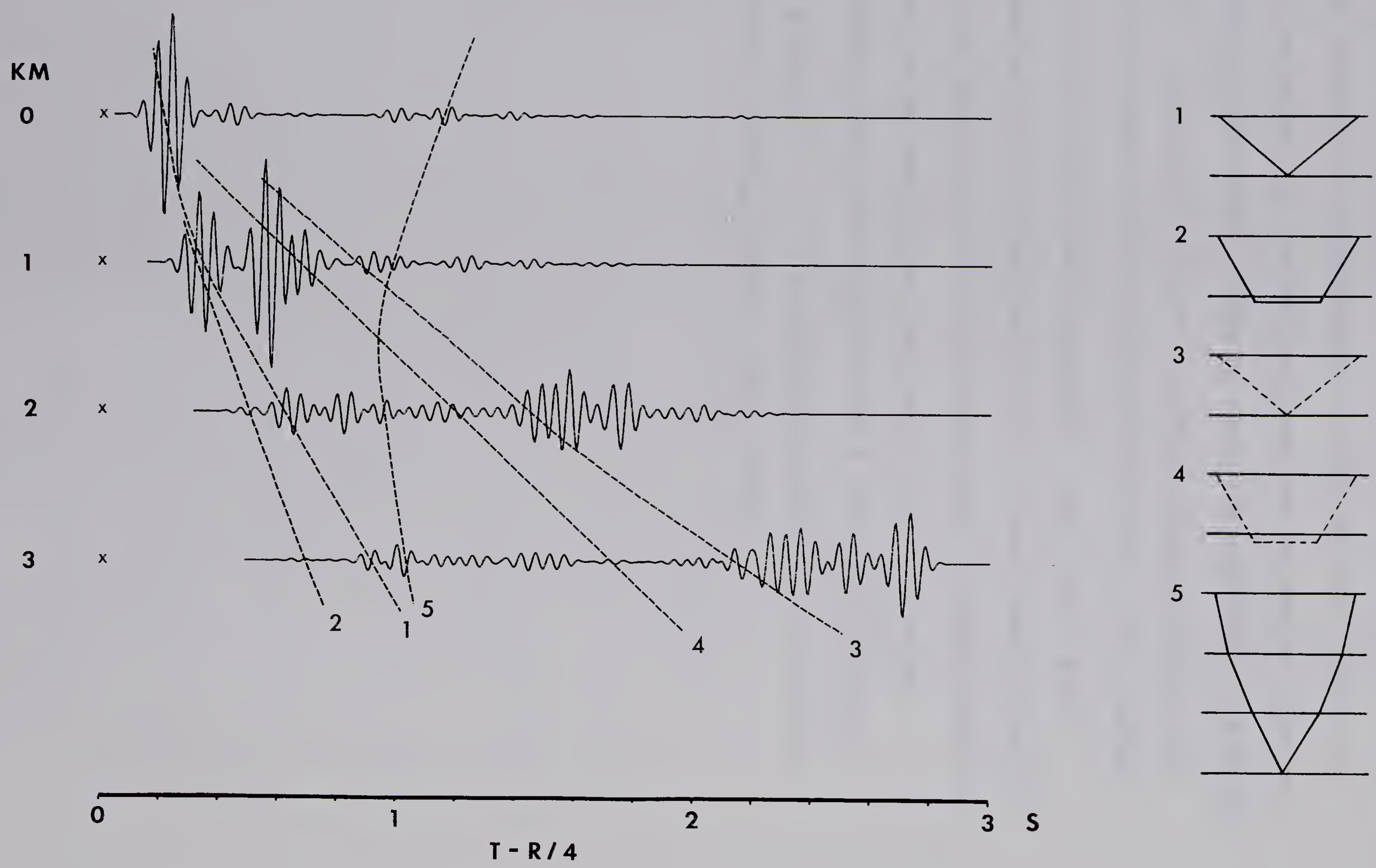


Figure 16 Synthetic seismograms for the homogeneous-layers approximation to the Mississippi model. The axis displays reduced time where R is the epicentral distance and a reducing velocity of 4 km/sec has been used.

MISSISSIPPI MØDEL : HØMØ. SCALING FACTOR: 0.4



time of primary reflections are preserved. This is standard practice in seismic exploration. Thus we see that the general character of the seismograms are somewhat similar. The dynamic properties, however, are considerably different. In general, approximation of an inhomogeneous medium by homogeneous layers is likely to be good only if a large number of thin layers are employed. Even then, good approximation is attained only for a given range of source frequencies and epicentral distances and a large number of rays need to be considered (Muller 1970). To study continuously refracted rays in media with continuously varying properties, it is much more efficient to employ the extended ART presented in this thesis.

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Appendix 1: Solution Near a Turning Point

Richards (1974) has shown that P and SV motion in vertically inhomogeneous media approximately decouple if the following potential representation is used:

$$\vec{u}(r,z,t) = \frac{1}{g(z)} \left\{ \nabla \left[\frac{g(z)}{\rho^{\frac{1}{2}}(z)} \Phi_p \; (r,z,\phi,t) \right] + \nabla\times\nabla\times \left[\frac{g(z)}{\rho^{\frac{1}{2}}(z)} \Phi_S \; (r,z,\phi,t) \; \vec{n}_z \right] \right\},$$

Here Φ_p, Φ_S are the P and SV displacement potentials and the scale factor g(z) is an arbitrary, smooth function of the elastic parameters. At high frequencies, these potentials satisfy the wave equations

$$\nabla^2\Phi(r,z,t) - \frac{1}{V^2(z)} \frac{\partial^2\Phi(r,z,t)}{\partial t^2} = 0$$

where v(z) and Φ may be P or SV wave velocities and potentials.

Using the Fourier and Hankel transforms defined in Chapter 2 and assuming azimuthal symmetry in the solution, we can reduce this equation to the Helmholtz equation:

$$\frac{d^2\Phi(p,z,\omega)}{dz^2} + \omega^2 q^2 \; \Phi \; (p,z,\omega) = 0. \qquad \text{A1.1}$$

We assume that q^2 can be expanded in a power series in z-z*, where z* is the turning point, and that in the neighborhood of z* we need consider only the linear term in the

expansion. Then

$$q^2 \simeq a(z - z^*) \qquad \text{A1.2}$$

where

$$a = \left| \frac{1}{V^3} \frac{dv}{dz} \right|_{z^*} \qquad \text{A1.3}$$

This assumption is invalid only in the special cases where the turning point coincides with an extremum of the velocity function and higher order terms must be used (Morse and Feshbach 1953, section 9-3).

Using this approximation and the substitution

$$y = (\omega a)^{1/3} (z^* - z) \qquad \text{A1.4}$$

we can convert A1.1 to the Stokes equation

$$\frac{d^2\Phi}{dy^2} = y\Phi$$

Solutions of this equation are well known (Budden 1961, Ch. 15). A physically meaningful solution for the displacement potential must be proportional to the Airy function Ai(y). If we consider the region just above the turning point, i.e. $z > z^*$ or $y < 0$, then for high frequencies, the solution has the form

$$\Phi(p,z,\omega) = C(\omega,z)\left[e^{i\frac{2}{3}|y|^{\frac{3}{2}}} + e^{i\frac{\pi}{2} - i\frac{2}{3}|y|^{\frac{3}{2}}}\right]$$

where C is an arbitrary function. With the help of A1.2 to A1.4 this in turn can be written as

$$\Phi(p,z,\omega) = C(\omega,z)\left[e^{i\omega\int_{z^*}^{z} q(\xi)d\xi} + e^{i\frac{\pi}{2} - i\omega\int_{z^*}^{z} q(\xi)d\xi}\right]$$

This has the same form as the WKBJ solution except for the $\exp(i\pi/2)$ phase factor associated with the upgoing (totally reflected) wave. Thus the solution near a turning point corresponds to propagation along the ray path of ray theory. However, analogous to the problem of reflection at interfaces, a phase change of $\pi/2$ now takes place as the ray is reflected at the turning point.

Appendix 2: The Method of Stationary Phase in n-Dimensions

In this appendix we derive an asymptotic formula for a function $u(\omega)$ defined by the n-dimensional integral

$$u(\omega) = \int_D g(T)\, e^{-i\omega\Phi(T)}\, dT \qquad \text{A2.1}$$

where $T = (t_1, \ldots, t_n)$, and D is the domain of integration. The method to be used is the method of stationary phase in n-dimensions (Lewis 1964).

A stationary point of A2.1 is a point $W = (w_1, \ldots, w_n)$ for which

$$\frac{\partial\Phi}{\partial t_i} = 0 \qquad i = 1, \ldots, n$$

Expanding Φ about the stationary point yields

$$\Phi(T) \simeq \Phi(W) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(t_i - w_i)(t_j - w_j)\,\Phi_{ij}$$

where

$$\Phi_{ij} \equiv \frac{\partial^2\Phi(W)}{\partial t_i\,\partial t_j}$$

We introduce a matrix $\overline{\overline{B}}$ whose elements are

$$B_{ij} = \Phi_{ij}$$

and a linear coordinate transformation $T \rightarrow T'$ such that

the stationary point is the origin in the new coordinates and the matrix $\bar{\bar{B}}$ is diagonal, i.e.,

$$\Phi_{ij}(W') = \nu_i \delta_{ij}$$

where ν_i i=1, ..., n are the eigenvalues of $\bar{\bar{B}}$ and δ_{ij} is the Kronecker delta function.

In the new coordinates, then

$$\omega\Phi(T') = \omega\Phi(W') + \tfrac{1}{2}\sum_{i=1}^{n} \nu_i \alpha_i^2 \qquad \text{A2.2}$$

where

$$\alpha_i = \omega^{\frac{1}{2}} t_i'$$

Using A2.2 and the Jacobian of transformation

$$\frac{\partial(\alpha_1, \ldots, \alpha_n)}{\partial(t_1', \ldots, t_n')} = \omega^{\frac{n}{2}}$$

in A2.1 we obtain

$$u(\omega) \simeq \left(\frac{2\pi}{\omega}\right)^{\frac{n}{2}} \frac{g(W')}{\left|\prod_{j=1}^{n} \nu_j\right|^{\frac{1}{2}}} e^{-i\omega\Phi(W') - i\frac{\pi}{4}\sum_{k=1}^{n} \operatorname{sgn} \nu_k}$$

Now

$$\prod_{k=1}^{n} \nu_k = \det \bar{\bar{B}}\ (W)$$

is the determinant of the matrix $\bar{\bar{B}}$; and

$$\sum_{k=1}^{n} \operatorname{sgn} \nu_k = \operatorname{sig} \bar{\bar{B}}\ (W)$$

is the signature of the matrix $\bar{\bar{B}}$. Both these quantities are invariant under linear coordinate transformations. Therefore, in the original coordinates

$$u(\omega) \simeq \left(\frac{2\pi}{\omega}\right)^{\frac{n}{2}} \frac{g(W)}{|\det\bar{\bar{B}}(W)|^{\frac{1}{2}}} \; e^{-i\omega\Phi(W)-i\frac{\pi}{4}\operatorname{sig}\bar{\bar{B}}(W)}$$

Appendix 3: Contour Integral Representation of Airy Functions

Airy functions are solutions of the Stokes equation

$$\frac{d^2u}{dy^2} = yu \qquad \text{A3.1}$$

We wish to look for integral solutions of the form

$$u = \int_a^b e^{-iys}\, h(s)\, ds \qquad \text{A3.2}$$

where s is a complex variable and a,b are the endpoints of some contour to be determined. Substitution of A3.2 into A3.1 and integration by parts yields

$$ih(s)\, e^{-iys} \Big|_a^b + \int_a^b \left\{ s^2 h(s) - i\frac{dh}{ds} \right\} e^{-iys}\, ds = 0 \qquad \text{A3.3}$$

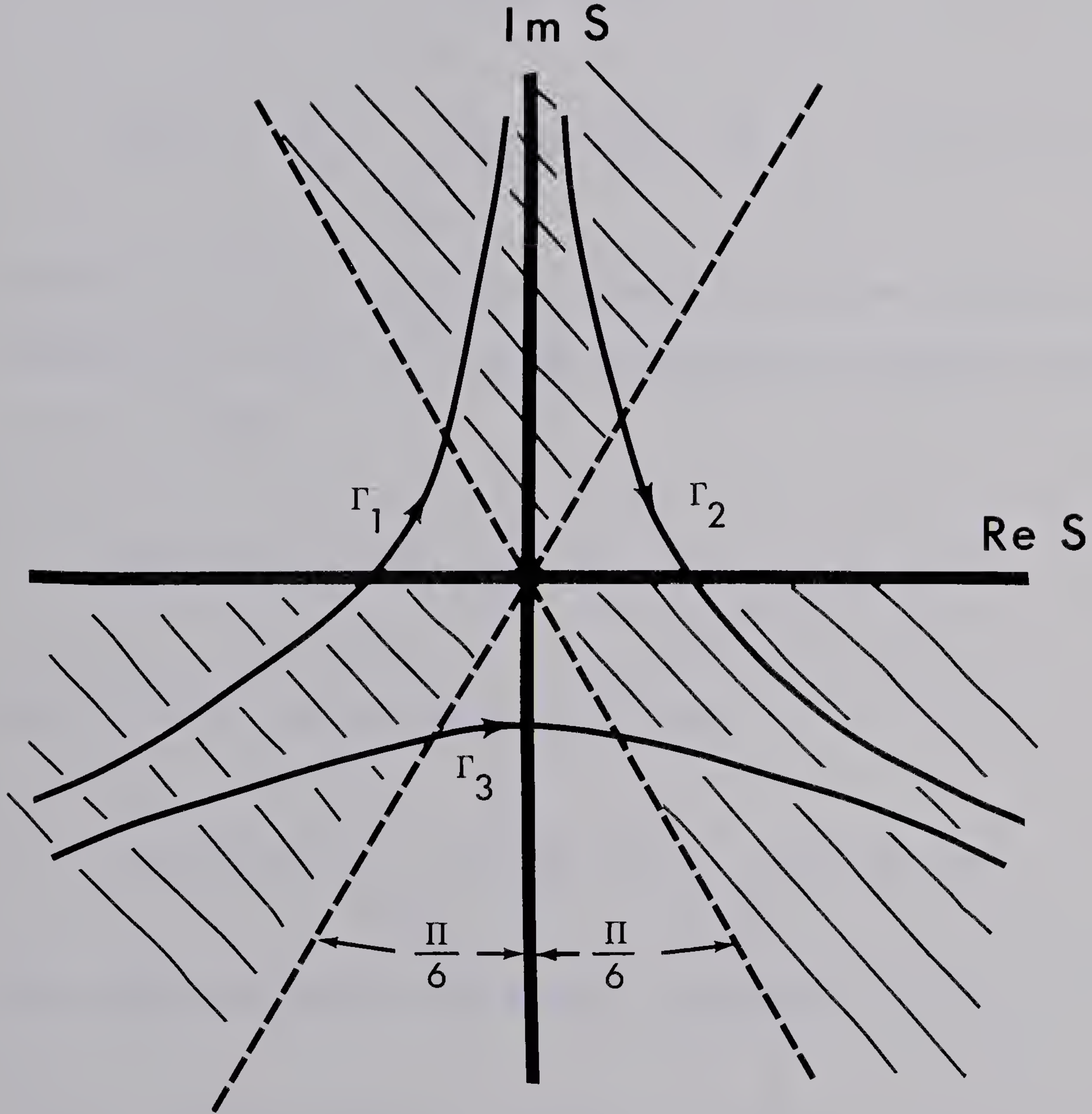
For the contours shown in Fig. 17, the first term of A3.3 vanishes at both limits provided the endpoints of the contours are kept within the shaded regions. This leads to solutions of the Stokes equation of the form

$$u_j = \int_{\Gamma_j} A_j\, Y(y,s)\, ds \quad ; \quad j = 1,2,3$$

for some arbitrary constants A_j and

$$Y(y,s) = e^{-i\left(\frac{s^3}{3} + ys\right)}$$

Figure 17 Contours for the integral A3.3.


Im S
Re S
Γ_1
Γ_2
Γ_3
$\frac{\Pi}{6}$
$\frac{\Pi}{6}$

In accordance with Budden (1961), Abramowitz and Stegun (1965) and others, two independent solutions, called the Airy functions, may be defined as follows:

$$\mathrm{Ai}(y) = \frac{1}{2\pi} \int_{\Gamma_3} Y(y,s)\, ds$$
$$\mathrm{Bi}(y) = \frac{i}{2\pi} \int_{\Gamma_2} Y(y,s)\, ds - \frac{i}{2\pi} \int_{\Gamma_1} Y(y,s)\, ds \qquad \text{A3.4}$$

Since $Y(y,s)$ is everywhere analytic in the s-plane, the contour Γ_3 in Fig. 17 can be deformed to coincide with $\Gamma_1 + \Gamma_2$. Then

$$2\pi \mathrm{Ai}(y) = \int_{\Gamma_1} Y(y,s)\, ds + \int_{\Gamma_2} Y(y,s)\, ds$$

But it also follows from A3.4 that

$$-i2\pi \mathrm{Bi}(y) = \int_{\Gamma_2} Y(y,s)\, ds - \int_{\Gamma_1} Y(y,s)\, ds$$

Therefore we obtain the useful formulae

$$\int_{\Gamma_1} Y(y,s)\, ds = \pi \left[\mathrm{Ai}(y) + i\mathrm{Bi}(y) \right]$$
$$\int_{\Gamma_2} Y(y,s)\, ds = \pi \left[\mathrm{Ai}(y) - i\mathrm{Bi}(y) \right]$$

Appendix 4: Ray Characteristics for Piece-wise Linear Interpolation

The velocity function is locally given by

$$v = v_i + g(z-z_i)$$

where

$$g = \frac{v_{i+1} - v_i}{z_{i+1} - z_i}$$

If we let

$$a_i = (1-p^2 v_i^2)^{1/2} \quad ; \quad a_{i+1} = (1-p^2 v_{i+1}^2)^{1/2}$$

then

$$r_i = \frac{a_i - a_{i+1}}{gp} \quad ; \quad t_i = \frac{1}{g} \ln \left[\frac{v_{i+1}}{v_i} \; \frac{1+a_i}{1+a_{i+1}} \right]$$

$$\frac{\partial r_i}{\partial p} = \frac{a_i - a_{i+1}}{gp^2 a_i a_{i+1}}$$

$$\frac{\partial^2 r_i}{\partial p^2} = \frac{V_{i+1}^2 - V_i^2}{g} \; \frac{V_{i+1}^2 \, a_i \, (2+a_i/a_{i+1}) + V_i^2 a_{i+1} (2+a_{i+1}/a_i)}{a_i^2 \, a_{i+1}^2 \, (a_i + a_{i+1})^2}$$

If the ray segment has a turning point at (v_{i+1}, z_{i+1}), then

$$\frac{\partial r_i}{\partial p} = \frac{-1}{p^2 g a_i}$$

$$\frac{\partial^2 r_i}{\partial p^2} = \frac{2-3p^2 V_i^2}{gp^3 a_i^3}$$

Appendix 5: Ray Characteristics for Smoothed Spline Interpolation v=v(z)

The velocity function is locally given by

$$\eta = b_o + b_1\tilde{z} + b_z\tilde{z}^2 + b_3\tilde{z}^3$$

where

$$\eta = \frac{1}{v} \quad ; \quad \tilde{z} = z - z_i$$

Then, letting $q = (\eta^2 - p^2)^{\frac{1}{2}}$,

$$r_i = \int_{z_i}^{z_{i+1}} \frac{p}{q}\, dz \quad ; \quad t_i = \int_{z_i}^{z_{i+1}} \frac{\eta^2}{q}\, dz$$

$$\frac{\partial r_i}{\partial p} = \int_{z_i}^{z_{i+1}} \frac{\eta^2}{q^3}\, dz \quad ; \quad \frac{\partial^2 r_i}{\partial p^2} = \int_{z_i}^{z_{i+1}} \frac{3p\eta^2}{q^5}\, dz$$

If the ray segment has a turning point at (r_{i+1}, z_{i+1}), a new variable of integration must be used to avoid the singularity of z_{i+1}.

Letting

$$y^2 = \eta - p$$

$$f = \frac{d\eta}{dz} \quad ; \quad g = \frac{d^2\eta}{dz^2}$$

$$h = (\eta + p)^{\frac{1}{2}} \quad ; \quad u_i = (\eta_i - p)^{\frac{1}{2}}$$

then

$$r_i = \int_o^{u_i} \frac{-2p}{fh}\, dy \quad ; \quad t_i = \int_o^{u_i} \frac{-2\eta^2}{fh}\, dy$$

$$\frac{\partial r_i}{\partial p} = \left[\frac{\eta}{fg}\right]_{z_i} - \int_o^{u_i} \left[1 - \frac{\eta g}{f}\right] \frac{2}{fh}\, dy$$

$$\frac{\partial^2 r_i}{\partial p^2} = \left\{ \frac{\eta^2}{pfg} \left[\frac{\eta}{q^2} - \frac{g}{f^2}\right] \right\}_{z_i}$$

$$+ \int_o^{u_i} \left[2g + \frac{\eta}{f} \frac{d^3\eta}{dz^3} - 3\eta \left(\frac{g}{f}\right)^2\right] \frac{2\eta}{pf^3 h}\, dy$$

Appendix 6: Ray Characteristics for Smoothed Spline Interpolation z=z(V)

The velocity-depth function is locally given by

$$z = b_o + b_1\tilde{V} + b_2\tilde{V}^2 + b_3\tilde{V}^3$$

where $\tilde{V} = V - V_i$. Then, letting $a = (1-p^2V^2)^{\frac{1}{2}}$

$$r_i = \left[\frac{-a}{p}\left\{b_1 - b_2(2V_i - V) + 3b_3(V_i{}^2 - V_iV)\right\} + \frac{b_2 - 3b_3V_i}{p^2}\sin^{-1}pV + \frac{ab_3}{p^3}(a^2-3)\right]_{V_i}^{V_{i+1}}$$

$$t_i = \left[\tfrac{1}{2}(b_1 - 2b_2V_i + 3b_3V_i{}^2)\ \ell n\left(\frac{1-a}{1+a}\right) + \frac{2}{p}(b_2 - 3b_3V_i)\sin^{-1}pV - \frac{3ab_3}{p}\right]_{V_i}^{V_{i+1}}$$

$$\frac{\partial r_i}{\partial p} = \left[\frac{1}{ap^2}\left\{b_1 + 2b_2(V-V_i) + 3b_3\left(\frac{1}{p^2} + V_i(V_i - 2V)\right)\right\} + \frac{3}{p^4}ab_3 - \frac{2}{p^3}(b_2 - 3b_3V_i)\sin^{-1} pV\right]_{V_i}^{V_{i+1}}$$

$$\frac{\partial^2 r_i}{\partial p^2} = \left[\frac{1}{ap}\left\{\left(\frac{V^2}{a^2} - \frac{2}{p^2}\right)[b_1 + 2b_2(V-V_i) + 3b_3(\frac{1}{p^2} + V_i(V_i - 2V))] - \frac{18}{p^4}b_3(1-\tfrac{1}{2}p^2V^2) + \frac{2}{p^2}(b_2 - 3b_3V_i)\left(\frac{3a}{p}\sin^{-1}pV - V\right)\right\}\right]_{V_i}^{V_{i+1}}$$

If the ray segment has a turning point at (r_i^*, z_i^*), then

$$\frac{\partial r_i}{\partial p} = \frac{1}{p^2}\left\{\frac{1}{a_i}\left[b_1+6b_3\left(\frac{a_i}{p}\right)^2\right]+\frac{2}{p}\left(b_2-3b_3V_i\right)\left(\frac{\pi}{2}-\sin^{-1}pV_i\right)\right\}$$

$$\frac{\partial^2 r_i}{\partial p^2} = \frac{1}{pa_i}\left\{\left[\frac{2}{p^2}-\left(\frac{V_i}{a_i}\right)^2\right]\left[b_1+6b_3\left(\frac{a_i}{p}\right)^2\right]+\frac{12}{p^4}b_3\right.$$

$$\left.+\frac{2}{p^2}\left(b_2-3b_3V_i\right)\left[V_i+\frac{3a_i}{p}\left(\frac{\pi}{2}-\sin^{-1}pV_i\right)\right]\right\}.$$

For small values of p, the inverse square root in 5.2 may be expanded for small values of pv. Keeping only up to the quadratic term of the expansion the results are

$$r_i = p\left[\tfrac{1}{2}C_0V^2+\tfrac{1}{3}C_1V^3+\frac{V^4}{4}\left(C_2+\tfrac{1}{2}p^2C_0\right)+\frac{p^2}{2}\left(\tfrac{1}{5}C_1V^5+\tfrac{1}{6}C_2V^6\right)\right]_{V_i}^{V_{i+1}}$$

$$t_i = \left[C_0\ln V+C_1V+\frac{V^2}{2}\left(C_2+\tfrac{1}{2}p^2C_0\right)+\frac{p^2}{2}\left(\tfrac{1}{3}C_1V^3+\tfrac{1}{4}C_2V^4\right)\right]_{V_i}^{V_{i+1}}$$

$$\frac{\partial r_i}{\partial p} = \left[\tfrac{1}{2}C_0V^2+\tfrac{1}{3}C_1V^3+\frac{V^4}{4}\left(C_2+\tfrac{3}{2}C_0p^2\right)+\frac{3p^2}{2}\left(\tfrac{1}{5}C_1V^5+\tfrac{1}{6}C_2V^6\right)\right]_{V_i}^{V_{i+1}}$$

$$\frac{\partial^2 r_i}{\partial p^2} = pV^4\left[3\left(\frac{C_0}{4}+\tfrac{1}{5}C_1V\right)+\left(\tfrac{5}{4}p^2C_0+\tfrac{1}{2}C_2\right)V^2\right.$$

$$\left.+\frac{15}{2}p^2\left(\tfrac{1}{7}C_1V^3+\tfrac{1}{8}C_2V^4\right)\right]_{V_i}^{V_{i+1}}$$

where

$$C_0 = b_1 - 2b_2V_i + 3b_3V_i^2$$

$$C_1 = 2(b_2 - 3b_3V_i)$$

$$C_2 = 3b_3$$